Statistical theory for the Kardar-Parisi-Zhang equation in (1+1) dimensions

A. A. Masoudi,[1] F. Shahbazi,[1] J. Davoudi,[2,3] and M. Reza Rahimi Tabar[4,5,6]
[1]*Department of Physics, Sharif University of Technology, P.O. Box 11365-9161, Tehran, Iran*
[2]*Max-Planck-Institute for Complex Systems, Noethnitzer Strasse 38, 01187 Dresden, Germany*
[3]*Fachbereich Physik, Philipps Universitaet, Renthof 6, 35032 Marburg, Germany*
[4]*CNRS UMR 6529, Observatoire de la Côte d'Azur, Boîte Postal 4229, 06304 Nice Cedex 4, France*
[5]*Department of Physics, Iran University of Science and Technology, Narmak, Tehran 16844, Iran*
[6]*Institute of Applied Physics, IUST, P.O. Box 16845-163, Tehran, Iran*
(Received 14 June 2001; published 23 January 2002)

The Kardar-Parisi-Zhang (KPZ) equation in (1+1) dimensions dynamically develops sharply connected valley structures within which the height derivative *is not* continuous. We develop a statistical theory for the KPZ equation in (1+1) dimensions driven with a random forcing that is white in time and Gaussian-correlated in space. A master equation is derived for the joint probability density function of height difference and height gradient $P(h-\overline{h},\partial_x h,t)$ when the forcing correlation length is much smaller than the system size and much larger than the typical sharp valley width. In the time scales before the creation of the sharp valleys, we find the exact generating function of $h-\overline{h}$ and $\partial_x h$. The time scale of the sharp valley formation is expressed in terms of the force characteristics. In the stationary state, when the sharp valleys are fully developed, finite-size corrections to the scaling laws of the structure functions $\langle (h-\overline{h})^n(\partial_x h)^m\rangle$ are also obtained.

DOI: 10.1103/PhysRevE.65.026132 PACS number(s): 05.70.Ln, 68.35.Fx

I. INTRODUCTION

There has been a great deal of recent work on the formation, growth, and geometry of interfaces [1–5]. The dynamics of interfaces has turned out to be one of the most fascinating and challenging topics in theoretical nonequilibrium physics. There are two principal approaches for theoretically analyzing such problems. The first is based on computer simulations of discrete models and often provides useful links between theoretical analysis and experiments. The second approach's aim is to describe the dynamical process by a stochastic differential equation. This procedure neglects the short length-scale details but provides a coarse-grained description of the interface (that is suitable for characterizing the asymptotic scaling behavior). Theoretical modeling of growth processes started with the work of Edwards and Wilkinson [6]. They suggested that one might describe the dynamics of the height fluctuations by a simple linear stochastic equation. Kardar, Parisi, and Zhang (KPZ) [7] realized that there is a relevant term proportional to the square of the height gradient which represents a correction for lateral growth. Indeed, the KPZ equation is a prototype model for a system in which the interface growth is subjected to a random external flux of particles. The randomness is described by an annealed random noise, which mimics the random adsorption of molecules onto a surface. In the KPZ model [e.g., in the (1+1) dimension], the surface height field $h(x,t)$ of a one-dimensional substrate satisfies a stochastic random equation,

$$\frac{\partial h}{\partial t}-\frac{\alpha}{2}(\partial_x h)^2=\nu\partial_x^2 h+f(x,t), \tag{1}$$

where $\alpha\geqslant 0$ and f is a zero-mean, statistically homogeneous, white in time, and Gaussian process with covariance

$$\langle f(x,t)f(x',t')\rangle=2D_0\delta(t-t')D(x-x'), \tag{2}$$

where

$$D(x-x')=\frac{1}{\sqrt{\pi}\sigma}\exp\left(-\frac{(x-x')^2}{\sigma^2}\right), \tag{3}$$

and σ is the variance of $D(x-x')$. Typically, the correlation of forcing is considered as a δ function for mimicking the short-range correlation. We regularize the δ-function correlation by a Gaussian function. When the variance σ is much less than the system size, we would expect that the model would represent a short-range correlated forcing. So we would stress that our calculations are done for finite $\sigma\ll L$, where L is the system size. The average force on the interface is unimportant and may be removed from the equation of motion by a boost transformation. Every term in Eq. (1) involves a specific physical phenomenon contributing to the surface evolution. The parameters ν, α, and D_0 (and σ) describe the surface diffusive relaxation, nonlinear lateral growth, and the effective noise strength, respectively.

We consider a substrate of size L and define the mean height of a growing film and its roughness w by

$$\overline{h}(L,t)=\frac{1}{L}\int_{-L/2}^{L/2}dx\,h(x,t), \tag{4}$$

$$w(L,t)=[\langle (h-\overline{h})^2\rangle]^{1/2}, \tag{5}$$

where $\langle\,\rangle$ denotes an averaging over different realizations of the noise (samples). Starting from a flat interface (one of the possible initial conditions), it was conjectured by Family and Vicsek [8] that a scaling of space by a factor b and of time by a factor b^z (z is the dynamical scaling exponent) rescales the roughness w by a factor b^χ as follows:

$$w(bL,b^z t)=b^\chi w(L,t), \tag{6}$$

which implies that

$$w(L,t)=L^\chi f\left(\frac{t}{L^z}\right). \tag{7}$$

If for large t and fixed $L[(t/L^z)\to\infty]$ w saturates, then $f(x)\to\text{const}$ as $x\to\infty$. However, for fixed large L and $1\ll t \ll L^z$, one expects that correlations of the height fluctuation are set up only within a distance $t^{1/z}$ and thus must be independent of L. This implies that for $x\ll 1$, $f(x)\sim x^\beta$ with $\beta=\chi/z$. Thus dynamic scaling postulates that [9]

$$\begin{aligned} w(L,t)&\sim t^\beta, \quad 1\ll t\ll L^z,\\ &\sim L^\chi, \quad t\gg L^z. \end{aligned} \tag{8}$$

The roughness exponent χ and the dynamic exponent z characterize the self-affine geometry of the surface and its dynamics, respectively. Several time regimes can be distinguished in the time evolution of the surface roughness. They can be summarized as follows: for very early times, the noise term dominates since its contribution to the equation grows as the square root of time. In this time regime, the surface roughness grows as $w(t)\sim t^{1/2}$. For intermediate times, the linear term has the main contribution. The linear case $(\alpha=0)$ is the Edwards-Wilkinson model, for which one can easily find that the surface roughness behaves as $w(t)\sim t^{\beta_0}$, where the value of β_0 depends on the dimension of the substrate ($\beta_0=[(2-d)/4]$ for a d-dimensional surface). For later times, the contribution of the relevant nonlinear term becomes a dominant one and the surface roughness growth is characterized by the behavior $w(t)\sim t^\beta$. For very late times and finite substrate length L, the roughness saturates as $w(t\to\infty,L)\sim L^\chi$. Of course, in an experiment or in a numerical simulation the transition between the different regimes is not sharp and different crossover behaviors can be observed. Galilean invariance implies the relation $\chi+z=2$ independent of dimension [10,11]. It means that there is only one independent exponent in the KPZ dynamics. In the one-dimensional substrates a fluctuation-dissipation theorem yields exactly $z=\frac{3}{2}$, $\chi=\frac{1}{2}$, and $\beta=\frac{1}{3}$ [12]. In contrast to one dimension, the case $d\geq 2$ can only be attacked by approximative field-theoretic perturbative expansions [13–16]. It is well known that the effective coupling constant for the KPZ equation is $g=2\alpha^2 D_0/\nu^3$. Phase diagram information extracted from the renormalization-group flow indicates that $d=2$ plays the role of a lower critical dimension. For $d\leq 2$, the Gaussian fixed point $(\alpha=0)$ is infrared-unstable, and there is a crossover to the stable strong-coupling fixed point. For $d>2$, a third fixed point exists, which represents the roughening transition. It is unstable and lies between the Gaussian and strong-coupling fixed points, which are now both stable. Only the critical exponents of the strong-coupling regime ($g\to\infty$ or $\nu\to 0$) are known in $(1+1)$ dimensions and their values in higher dimensions as well as properties of the roughening transition have been known only numerically [17–23] and by the various approximative schemes [24–32].

The theoretical richness of the KPZ model is partly due to close relationships with other areas of statistical physics. It is shown that there is a mapping between the equilibrium statistical mechanics of a two-dimensional smectic-A liquid crystal onto the nonequilibrium dynamics of the $(1+1)$-dimensional stochastic KPZ equation [33]. It has been shown in [34] that one can map the kinetics of the annihilation process $A+B\to 0$ with driven diffusion onto the $(1+1)$-dimensional KPZ equation. Also the KPZ equation is closely related to the dynamics of a sine-Gordon chain [35], the driven-diffusion equation [36,37], high-T_c superconductor [38], directed paths in the random media [39–52] and charge-density waves [53], dislocations in disordered solids [3], the formation of a large-scale structure in the universe [54–57], Burgers turbulence [58–85,90], etc.

As already mentioned, the main difficulty with the KPZ equation is that it is controlled, in all dimensions, by a strong disorder (or strong-coupling) fixed point and efficient tools are missing to calculate the exponents and other universal properties, e.g., scaling functions, amplitudes, etc. Despite the fact that in one dimension the exponents are known, many properties, including the probability density function (PDF) of the height of a growing interface, have so far been measured only in numerical simulations. Recently, Derrida and Lebowitz have shown that for one particular model of the KPZ class, the asymmetric exclusion process (ASEP), the distribution of the displacement of particles could be calculated for a finite geometry by the Bethe ansatz [86,87]. It is proved in [86,87] that the distribution of deviation y of the average current is skewed and has the following asymptotic [88]:

$$\begin{aligned} P(y)&\sim \exp(-Ay^{5/2}), \quad y\to+\infty,\\ &\sim \exp(-B|y|^{3/2}), \quad \to-\infty. \end{aligned} \tag{9}$$

More recently, Prähofer and Spohn mapped the polynuclear growth model (PNG) onto random permutations, where the height is the length of the longest increasing subsequence of such a permutation, and thereby onto Gaussian random matrices. Hence they succeeded to obtain an analytic expression for a certain scaling distribution, which led to an understanding of how the self-similar height fluctuations depend on the initial conditions [94].

In this paper, we are interested in the statistical properties of the KPZ equation in the strong-coupling limit $(\upsilon\to 0)$. The limit is singular, i.e., the surface develops sharp valleys. Therefore, starting with a flat surface after a finite time scale, t_c, the sharp valley singularities are dynamically developed. In the singular points (sharp valleys), the spatial derivative of the $h(x,t)$ is not continuous. Hence the limit of $\nu\to 0$ is not singular for $t<t_c$, and we can ignore the diffusion term, while after developing the singularities the diffusion term has a finite contribution in the PDF of height fluctuations. Inspired by the methods proposed recently in the works of Weinan E and Vanden Eijnden [73], we develop a statistical

method to describe the moments of height difference and height gradient of height field $h(x,t)$. We derive a master equation for the joint PDF of the height difference $(h-\overline{h})$ and height gradient $\partial_x h$, $P((h-\overline{h}),\partial_x h)$, for a given g or diffusion constant ν. We will consider two different time scales in the limit of $\nu\to 0$: (i) early stages before developing the sharp valley singularities, and (ii) an established stationary state comprised of fully developed sharp valley singularities. In the regime (i), ignoring the relaxation term in the equation of the joint PDF when $\nu\to 0$, we determine the exact generating function of joint moments of height and height gradient fields. The realizability condition for the resulting joint PDF sheds light on the time scale of the sharp valley formation. In contrast, the limit $\nu\to 0$ is singular in the regime (ii), leading to an unclosed term (relaxation term) in the PDF equation. However, we show that the unclosed term can be expressed in terms of statistics of some quantities defined on the singularities (sharp valleys). Identifying each sharp valley in position y_0 with three quantities, namely the gradient of h in the position y_{0+}, y_{0-} and its height from $\overline{h}$, we determine the dynamics of these quantities. In both regimes, all the moments $\langle (h-\overline{h})^n(\partial_x h)^m\rangle$ for a given n and m are found. In the regime (ii) we will prove that in leading order, when $L\to\infty$, fluctuation of the height field is not intermittent and also we succeed in giving the analytic form of the amplitudes of all the structure functions. In addition, the scaling behavior and the amplitudes of all the correction terms due to the finite-size effect are calculated.

The paper is organized as follows. In Sec. II, we derive the master equation for the joint PDF of height differences and height gradients for given diffusion υ. We convert the height PDF, i.e., $P(h-\overline{h},t)$, evolution equation consequent to a Fokker-Planck equation for an arbitrary given diffusion constant. In Sec. III, we consider the limit of $\upsilon\to 0$ of the master equation in the time scales in which there is no singularity in the surface (before developing the sharp valleys). We determine the exact and explicit expression of the generating function for the moments $\langle (h-\overline{h})^n(\partial_x h)^m\rangle$ for given n and m. In Sec. IV, we consider the master equation in the limit $\nu\to 0$ and consequently when the singularities are fully developed. In this regime, the relaxation term has a finite contribution in the master equation. Using the methods introduced in [73], we prove that the unclosed term can be written in terms of quantities which are defined on sharp valleys where $\partial_x h$ is discontinuous. Also in this section, we determine the relation between the density of sharp valleys and the forcing variance $D_{xx}(0)$ in detail. In Sec. V, we derive the moments of height fluctuation in the stationary state and show that the PDF of $(h-\overline{h})$ is strongly asymmetric, and we prove that up to leading order the nth moments of $(h-\overline{h})$, i.e., $\langle (h-\overline{h})^n\rangle$, can be written in terms of the second-order moment of height fluctuation in a nonintermittent way. We determine all of the moments and show that the amplitudes of moments $\langle (h-\overline{h})^n(\partial_x h)^m\rangle$ can be written in terms of characteristics of singularities. We also derive the finite-size effect on the moments of height differences and determine the amplitudes of all correction terms. We have left the details of the calculations to Appendixes A–E.

II. THE MASTER EQUATION FOR HEIGHT DIFFERENCE AND HEIGHT GRADIENT

In this section, we consider the $(1+1)$-dimensional KPZ equation and derive the master equation to describe the joint-PDF of height difference and height gradient, i.e., $P(h-\overline{h},\partial_x h)$, for given ν and α. It is shown that the equation for the joint PDF is not closed due to the *linear* term $\nu\partial^2 h$. The PDF of height difference is related to the joint PDF, $P(h-\overline{h},\partial_x h,t)$, by the relation $P(h-\overline{h},t)=\int_{-\infty}^{\infty}P(h-\overline{h},\partial_x h,t)d(\partial_x h)$. We show that $P(h-\overline{h},t)$ satisfies a Fokker-Planck equation and we write down the explicit expression of drift and diffusion coefficient $D^{(1)}$ and $D^{(2)}$. It is shown that the drift and the diffusion coefficients can be written in terms of the conditional average $\langle(\partial_x h)^2|h-\overline{h}\rangle$.

We consider a one-dimensional substrate of length L and a surface of height field $h(x,t)$ and its gradient $\partial_x h(x,t)$ at time t. The $(1+1)$-dimensional KPZ equation governed on the height field $h(x,t)$ is defined in Eq. (1), while $u(x,t)=-\partial_x h(x,t)$ is a solution of the so-called Burgers equation,

$$u_t+\alpha u u_x=\nu u_{xx}-f_x(x,t), \tag{10}$$

where the covariance of f is given by Eqs. (2) and (3). To investigate the statistical properties of Eqs. (10) and (1), let us define the generating function $Z(\lambda,\mu,x,t)$ as

$$Z(\lambda,\mu,x,t)=\langle \exp(-i\lambda[h(x,t)-\overline{h}]-i\mu u(x,t))\rangle. \tag{11}$$

It follows from Eqs. (10) and (1) that the generating function Z is a solution of the following equation:

$$\begin{aligned}Z_t={}&i\gamma(t)\lambda Z-i\lambda\frac{\alpha}{2}Z_{\mu\mu}-\lambda^2 k(0)Z+\frac{i\alpha\lambda+\nu\lambda^2}{\mu}Z_\mu\\&-\frac{\nu\lambda}{\mu}Z_x-i\alpha\mu\left(\frac{Z_x}{\mu}\right)_\mu+\mu^2k_{xx}(0)Z\\&-i\mu\nu\langle u_{xx}\exp(-i\lambda(h-\overline{h})-i\mu u)\rangle,\end{aligned} \tag{12}$$

where $k(x-x')=2D_0D(x-x')$, $\gamma(t)=\overline{h}_t$, $k(0)=D_0/\sqrt{\pi}\sigma$, and $k_{xx}(0)=-2D_0/\sqrt{\pi}\sigma^3$. To derive Eq. (12), we have used the following identities:

$$\langle h_{xx}\exp(-i\lambda[h(x,t)-\overline{h}]-i\mu u(x,t))\rangle=\frac{-i}{\mu}\{Z_x+\lambda\partial_\mu Z\}, \tag{13}$$

$$\langle f(x,t)\exp(-i\lambda[h(x,t)-\overline{h}]-i\mu u(x,t))\rangle=-i\lambda k(0)Z, \tag{14}$$

and

$$\langle f_x(x,t)\exp(-i\lambda[h(x,t)-\overline{h}]-i\mu u(x,t))\rangle=-i\mu k_{xx}(0)Z, \tag{15}$$

where we have used the fact that $D_x(0)=0$. Evidently the last term in Eq. (12) is not closed. Assuming statistical homogeneity $(Z_x=0)$, we have

$$-i\mu Z_t=\gamma\lambda\mu Z-\frac{\alpha}{2}\lambda\mu Z_{\mu\mu}+i\lambda^2\mu k(0)Z -i\mu^3 k_{xx}(0)Z-i(\nu\lambda^2+i\alpha\lambda)Z_\mu -\mu^2\nu\langle u_{xx}\exp(-i\lambda\tilde{h}(x,t)-i\mu u(x,t))\rangle, \tag{16}$$

where $\tilde{h}(x,t)=h(x,t)-\overline{h}$. Defining $P(\tilde{h},u,t)$ as the joint probability density function of $\tilde{h}$ and u, one can construct $P(\tilde{h},u,t)$ in terms of the generating function Z as

$$P(\tilde{h},u,t)=\int\int\frac{d\lambda}{2\pi}\frac{d\mu}{2\pi}\exp(i\lambda\tilde{h}+i\mu u)Z(\lambda,\mu,t). \tag{17}$$

It follows from Eqs. (17) and (16) that $P(\tilde{h},u,t)$ satisfies the following equation:

$$-P_{ut}=-\gamma P_{\tilde{h}u}-\frac{\alpha}{2}(u^2P)_{\tilde{h}u}-\alpha(uP)_{\tilde{h}}-k(0)P_{\tilde{h}\tilde{h}u} +k_{xx}(0)P_{uuu}-\nu(uP)_{\tilde{h}\tilde{h}}-\nu\int\int\frac{d\lambda}{2\pi}\frac{d\mu}{2\pi}\exp(i\lambda\tilde{h} +i\mu u)\mu^2\langle u_{xx}\exp(-i\lambda\tilde{h}(x,t)-i\mu u(x,t))\rangle. \tag{18}$$

Now we can rewrite the last term in Eq. (18) as

$$\nu\int\int\frac{d\lambda}{2\pi}\frac{d\mu}{2\pi}\exp(i\lambda\tilde{h}+i\mu u)\mu^2\langle u_{xx}(x,t) \times\exp(-i\lambda\tilde{h}(x,t)-i\mu u(x,t))\rangle =-\nu\langle u_{xx}(x,t)\delta(\tilde{h}(x,t)-\tilde{h})\delta(u(x,t)-u)\rangle =-\nu\{\langle u_{xx}|u,\tilde{h}\rangle P(u,\tilde{h},t)\}_{uu}, \tag{19}$$

where $\langle\ |u,\tilde{h}\rangle$ denotes the average conditional on a given $u,\tilde{h}$. Therefore, using Eq. (19), it follows that $P(\tilde{h},u,t)$ satisfies the following equation:

$$-P_{ut}=-\gamma P_{\tilde{h}u}-\frac{\alpha}{2}(u^2P)_{\tilde{h}u}-\alpha(uP)_{\tilde{h}}-k(0)P_{\tilde{h}\tilde{h}u} +k_{xx}(0)P_{uuu}-\nu(uP)_{\tilde{h}\tilde{h}}+\nu\{\langle u_{xx}|u,\tilde{h}\rangle P(u,\tilde{h},t)\}_{uu}. \tag{20}$$

This equation is exact for a given ν and clearly the trace of the diffusion term leads again to an unclosed equation for $P(\tilde{h},u,t)$. Obtaining the functional form of the conditional averaging $\langle u_{xx}|u,\tilde{h}\rangle$ is one of the major difficulties in the formulation. From Eq. (18) we see that $P(u,\tilde{h})=P(-u,\tilde{h})$, which results in

$$\langle u|\tilde{h}\rangle=0 \tag{21}$$

for any given ν. In fact, for a given h the average of height gradient, u, is consequently zero. The identity proposed by Eq. (21) is not restricted to any limiting asymptotic and is true in all regimes of the dynamical evolution of the surface. Also Eq. (20) allows us to determine an evolution equation for $P(h-\overline{h})$. Doing so, we multiply Eq. (20) by u and integrate over u from $-\infty$ to $+\infty$, from which we get

$$\partial_t P(\tilde{h},t)=\frac{\partial}{\partial\tilde{h}}\left\{\left(\frac{\alpha}{2}(\langle u^2\rangle-\langle u^2|\tilde{h}\rangle)\right)P(\tilde{h},t)\right\} +\frac{\partial^2}{\partial\tilde{h}^2}\{[k(0)-\nu\langle u^2|\tilde{h}\rangle]P(\tilde{h},t)\}, \tag{22}$$

where $\tilde{h}=h-\overline{h}$ and the relation $\gamma=\alpha/2\langle u^2\rangle$ is used. This is a Fokker-Planck (FP) equation, describing the time evolution of $P(\tilde{h},t)$. The drift coefficient in the FP equation is

$$D^{(1)}=-\frac{\alpha}{2}(\langle u^2\rangle-\langle u^2|\tilde{h}\rangle) \tag{23}$$

and the diffusion coefficient reads

$$D^{(2)}=k(0)-\nu\langle u^2|\tilde{h}\rangle \tag{24}$$

Evidently, to obtain $P(h-\tilde{h})$ one should know the conditional average $\langle u^2|\tilde{h}\rangle$. The equation has the following stationary solution:

$$P_{\rm stat}(\tilde{h})=\frac{N}{D^{(2)}}\exp\left\{\int_{\tilde{h}_0}^{\tilde{h}}d\tilde{h}'D^{(1)}(\tilde{h}')/D^{(2)}(\tilde{h}')\right\}, \tag{25}$$

where N is the normalization coefficient. Therefore, to derive the moments of height difference $h-\overline{h}$, i.e., $\langle(h-\overline{h})^n\rangle$, we need the conditional averaging $\langle u^2|\tilde{h}\rangle$. The simplified picture given by this equation indicates that all the knowledge to obtain the behavior of PDF is buried in the functional form of one conditional average, i.e., $\langle u^2|\tilde{h}\rangle$. Although simple, it is clear that the conditional average $\langle u^2|\tilde{h}\rangle$ would have a nontrivial dependence on ν and L in the limit of $\nu\to 0$. Instead of following this strategy, however, in the next section we follow another direct way of extracting the moments of height difference $(h-\overline{h})$ in the strong-coupling limit, i.e., $\nu\to 0$.

III. THE JOINT CORRELATIONS OF HEIGHT DIFFERENCE AND HEIGHT GRADIENT BEFORE SHARP VALLEY FORMATION

When σ is finite, the very existence of the nonlinear term in the KPZ equation leads to the development of the sharp valley singularities in a *finite time* and in the strong-coupling limit $(\nu\to 0)$. In one dimension, the system is already in the strong-coupling regime, so starting from any finite value of ν at large time, the system develops sharp valley singularities

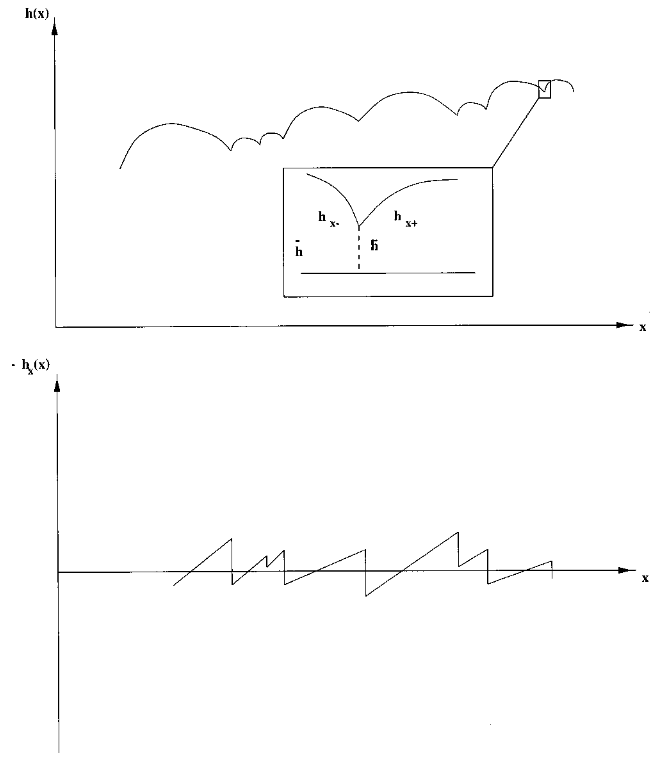

FIG. 1. In the upper graph the sharp valley solution in the KPZ equation are demonstrated while in the lower one the corresponding shock structures in the Burgers equation are sketched. The variables characterizing a sharp valley, namely h_{x-}, h_{x+}, and $\tilde{h}$, are shown.

(Fig. 1). Therefore, one would distinguish between different time regimes before and after the sharp valley formation. Starting from a flat initial condition, i.e., $h(x,0)=0$, $u(x,0)=0$ for a one-dimensional surface, which evolution is given by the inviscid KPZ equation ($\nu\to 0$), we know that after a finite time the derivative of function $h(x,t)$ becomes singular. After this time scale, the diffusion term is important, but we can neglect this term before the appearance of the singularities. So the equation governing the evolution of the generating function, $Z(\mu,\lambda,t)$, before the creation of the sharp valleys is given by

$$Z_t = i\gamma(t)\lambda Z - i\lambda\frac{\alpha}{2}Z_{\mu\mu} - \lambda^2 k(0)Z + \frac{i\alpha\lambda}{\mu}Z_\mu - i\alpha\mu\left(\frac{Z_x}{\mu}\right)_\mu + \mu^2 k_{xx}(0)Z, \tag{26}$$

in which we have assumed the statistical homogeneity ($Z_x=0$). Now we need the γ, which is given by $\gamma=\overline{h_t}$. Using Eq. (1), we get

$$\gamma(t)=\overline{h_t}=\frac{\alpha}{2}\langle u^2\rangle. \tag{27}$$

To evaluate $\langle u^2\rangle$, we set $\lambda=0$ in Eq. (26) and find

$$Z_t=\mu^2 k_{xx}(0)Z, \tag{28}$$

for which, considering $Z(\mu,0)=1$ as the initial condition, its solution is

$$Z(\mu,t)=\exp(\mu^2 k_{xx}(0)t). \tag{29}$$

On the other hand, by definition we have $\langle u^2\rangle = -([\partial^2 Z(\mu,t)]/\partial\mu^2)_{\mu=0}$. So before the creation of the singularities, the second moment of height gradient behaves as

$$\langle u^2\rangle = -2k_{xx}(0)t, \tag{30}$$

so consequently,

$$\gamma(t)=-\alpha k_{xx}(0)t. \tag{31}$$

Inserting Eq. (31) into Eq. (26) gives

$$\frac{\partial}{\partial t}Z(\mu,\lambda,t) = -i\alpha\lambda\frac{\partial^2}{\partial\mu^2}Z(\mu,\lambda,t) + i\alpha\frac{\lambda}{\mu}\frac{\partial}{\partial\mu}Z(\mu,\lambda,t) + [\mu^2 k_{xx}(0) - i\alpha k_{xx}(0)t\lambda - \lambda^2 k(0)] \times Z(\mu,\lambda,t). \tag{32}$$

We solve Eq. (32) with the initial condition $Z(\mu,\lambda,0)=1$, from which by expanding the generating function in powers of λ and μ we can obtain the moments $\langle(h-\overline{h})^n\rangle$, $\langle u^n\rangle$, and $\langle(h-\overline{h})^n u^m\rangle$. Changing the variable μ to $y=\mu^2$, converts Eq. (32) to the following equation:

$$\frac{\partial}{\partial t}Z(y,\lambda,t) = -2i\alpha\lambda y\frac{\partial^2}{\partial y^2}Z(y,\lambda,t) + i\alpha\lambda\frac{\partial}{\partial y}Z(y,\lambda,t) + [yk_{xx}(0) - i\alpha k_{xx}(0)t\lambda - \lambda^2 k(0)]Z(y,\lambda,t). \tag{33}$$

Introducing the Fourier transform of $Z(y,\lambda,t)$ with respect to y as $Q(q,\lambda,t)$, it is simple to get the following evolution equation satisfied by the Fourier transform:

$$\frac{\partial}{\partial t}Q(q,\lambda,t) = 2\alpha\lambda q^2\frac{\partial}{\partial q}Q(q,\lambda,t) + 5\alpha\lambda qQ(q,\lambda,t) - ik_{xx}(0)\frac{\partial}{\partial q}Q(q,\lambda,t) - i\alpha k_{xx}(0)t \times\lambda Q(q,\lambda,t) - \lambda^2 k(0)Q(q,\lambda,t), \tag{34}$$

with the initial condition

$$Q(q,\lambda,0)=\frac{1}{2\pi}\int e^{iyq}dy=\delta(q). \tag{35}$$

Equation (34) is a first-order partial differential equation which can be solved by the method of characteristics. The general solution of Eq. (34) is written as

$$Q(q,\lambda,t)=g\left(\lambda,\frac{1}{2}\frac{2tk_{xx}(0)\alpha\lambda+\sqrt{-2ik_{xx}(0)\alpha\lambda}\tanh^{-1}\left(q\sqrt{-\frac{2i\alpha\lambda}{k_{xx}(0)}}\right)}{k_{xx}(0)\alpha\lambda}\right)$$
$$\times\exp\left\{-1/2\int_0^q\left(\frac{10\alpha\lambda s+i\tanh^{-1}\left(s\sqrt{\frac{-2i\alpha\lambda}{k_{xx}(0)}}\right)\sqrt{-2ik_{xx}(0)\alpha\lambda}}{2\alpha\lambda s^2-ik_{xx}(0)}\right.\right.$$
$$\left.\left.+\frac{2i\alpha k_{xx}(0)t\lambda-2\lambda^2k(0)-i\tanh^{-1}\left(q\sqrt{\frac{-2i\alpha\lambda}{k_{xx}(0)}}\right)\sqrt{-2ik_{xx}(0)\alpha\lambda}}{2\alpha\lambda s^2-ik_{xx}(0)}\right)ds\right\}, \tag{36}$$

where g is an arbitrary function of its arguments. Imposing the initial condition, given by Eq. (35), and introducing ω as

$$\omega=\frac{1}{2}\frac{\tanh^{-1}\left(q\sqrt{\frac{-2i\alpha\lambda}{k_{xx}(0)}}\right)\sqrt{-2ik_{xx}(0)\alpha\lambda}}{k_{xx}(0)\alpha\lambda}, \tag{37}$$

we obtain

$$g(\lambda,\omega)=\delta\left(-\frac{1}{2}\sqrt{2}\sqrt{\frac{ik_{xx}(0)}{\alpha\lambda}}\tanh[\sqrt{2ik_{xx}(0)\alpha\lambda}\,\omega]\right)\exp\left[1/2\int_0^{-1/2\sqrt{2}\sqrt{ik_{xx}(0)/\alpha\lambda}\tanh[\sqrt{2ik_{xx}(0)\alpha\lambda}\,\omega]}\right.$$
$$\left.\times\left(\frac{10\alpha\lambda s+i\tanh^{-1}\left(s\sqrt{\frac{-2i\alpha\lambda}{k_{xx}(0)}}\right)\sqrt{-2ik_{xx}(0)\alpha\lambda}-2i\omega k_{xx}(0)\alpha\lambda-2\lambda^2k(0)}{2\alpha\lambda s^2-ik_{xx}(0)}\right)ds\right], \tag{38}$$

from which $Q(q,\lambda,t)$ is obtained as

$$Q(q,\lambda,t)=g(\lambda,t+\omega)\exp\left[-1/2\int_0^q\frac{10\alpha\lambda s+i\tanh^{-1}\left(s\sqrt{\frac{-2i\alpha\lambda}{k_{xx}(0)}}\right)\sqrt{-2ik_{xx}(0)\alpha\lambda}}{2\alpha\lambda s^2-ik_{xx}(0)}\right.$$
$$\left.-\frac{2i\alpha\lambda k_{xx}(0)t-i\tanh^{-1}\left(q\sqrt{\frac{-2i\alpha\lambda}{k_{xx}(0)}}\right)\sqrt{-2ik_{xx}(0)\alpha\lambda}-2\lambda^2k(0)}{2\alpha\lambda s^2-ik_{xx}(0)}ds\right]. \tag{39}$$

Inverse Fourier transforming of the solution in Eq. (39) is straightforward, so after switching to variable μ we get the following solution for $Z(\mu,\lambda,t)$:

$$Z(\mu,\lambda,t)=\{1-\tanh^2[\sqrt{2ik_{xx}(0)\alpha\lambda}t]\}$$
$$\times\exp\left\{-\tfrac{5}{8}\ln\{1-\tanh^4[\sqrt{2ik_{xx}(0)\alpha\lambda}t]\}\right.$$
$$+\tfrac{5}{4}\tanh^{-1}\{\tanh^2[\sqrt{2ik_{xx}(0)\alpha\lambda}t]\}-\lambda^2k(0)t$$
$$-\tfrac{1}{16}\ln^2\left(\frac{1-\tanh[\sqrt{2ik_{xx}(0)\alpha\lambda}t]}{1+\tanh[\sqrt{2ik_{xx}(0)\alpha\lambda}t]}\right)$$
$$\left.-\tfrac{1}{2}i\mu^2\sqrt{\frac{2ik_{xx}(0)}{\alpha\lambda}}\tanh[\sqrt{2ik_{xx}(0)\alpha\lambda}t]\right\}. \tag{40}$$

Since we are interested in moments $\langle(h-\overline{h})^n\rangle$, setting $\mu=0$ in Eq. (40) and expanding the generating function in powers of λ enables us to obtain them all. For example, expanding up to $O(\lambda^7)$, it is easy to see that the first sixth order of moments behaves as follows:

$$\langle(h-\overline{h})^2\rangle=-\tfrac{1}{3}t[k_{xx}(0)^2\alpha^2t^3-6k(0)], \tag{41}$$

$$\langle(h-\overline{h})^3\rangle=-\tfrac{24}{45}k_{xx}(0)^3\alpha^3t^6, \tag{42}$$

$$\langle(h-\overline{h})^4\rangle=-\tfrac{101}{105}k_{xx}(0)^4\alpha^4t^8-4t^5k_{xx}(0)^2\alpha^2k(0)+12t^2k(0)^2, \tag{43}$$

$$\langle(h-\overline{h})^5\rangle=-[\tfrac{2288}{945}k_{xx}(0)^5\alpha^5t^{10}+\tfrac{32}{3}k_{xx}(0)^3\alpha^3t^7k(0)], \tag{44}$$

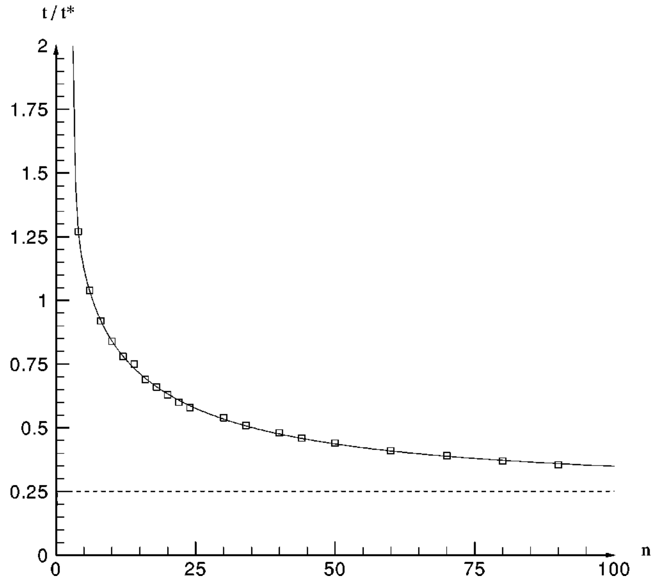

FIG. 2. Behavior of the time scales in which the moments $\langle(h-\tilde{h})^{2n}\rangle$ become negative in terms of *n*. The square points are calculated according to Eq. (40) while the solid line is the fitting curve asymptotically tending to $1/4(k(0)/\alpha^2 k_{xx}^2(0))^{1/3}$.

$$\langle(h-\overline{h})^6\rangle = -\frac{1}{10\,395}t^3[85\,783k_{xx}(0)^6\alpha^6t^9 + 299\,970k_{xx}(0)^4\alpha^4t^6k(0) + 623\,700k_{xx}(0)^2\alpha^2t^3k(0)^2 - 1\,247\,400k(0)^3]. \quad (45)$$

The important content of the exact forms derived above is that through it the time scale of sharp valley formation can be found. One should first check the realizability condition, i.e., $P(h-\overline{h},t)>0$. In fact, the above moment relations indicate that different even-order moments become *negative* in some distinct characteristic time scales. Closer inspection of the even-moment relations reveals that the higher the moments, the smaller are their characteristic time scales, so they asymptotically tend to $\frac{1}{4}t^*$ for very large even moments, where $t^*=[k(0)/\alpha^2k_{xx}^2(0)]^{1/3}$ (see Fig. 2). Therefore, we conclude that after this time the far tails of the probability distribution function start to become negative, which is reminiscent of sharp valley creation. This means that after the characteristic time scale $t_c=t/4^*$, one should also consider the contribution of the relaxation term in the limit of vanishing diffusion in order to find a realizable probability density function of height field. In other words, disregarding the diffusion term in the PDF equation is valid only up to the time scales in which the singularities are developed. Taking into account that $\alpha>0$, the odd-order moments are positive in time scales before the formation of sharp valleys. This means that the probability density $P(h-\overline{h},t)$ in this time regime is positively skewed. In Figs. 3–5, we have demonstrated the role of σ on the time scale of the creation of singularities. Substituting $k_{xx}(0)$ and $k(0)$ in the expression of t^* gives us $t^*=(\frac{1}{4})^{1/3}(\pi)^{1/6}D_0^{-1}\alpha^{-2/3}\sigma^{5/3}$. Hence the smaller the σ, the shorter the time scale of shock creation (see Figs. 3, 4, and 5) [93].

IV. THE EQUATION OF THE JOINT PDF OF HEIGHT DIFFERENCE AND HEIGHT GRADIENT IN THE STATIONARY STATE

Assuming the stationary state, we are interested in investigating the stationary solutions of Eq. (20) in the limit

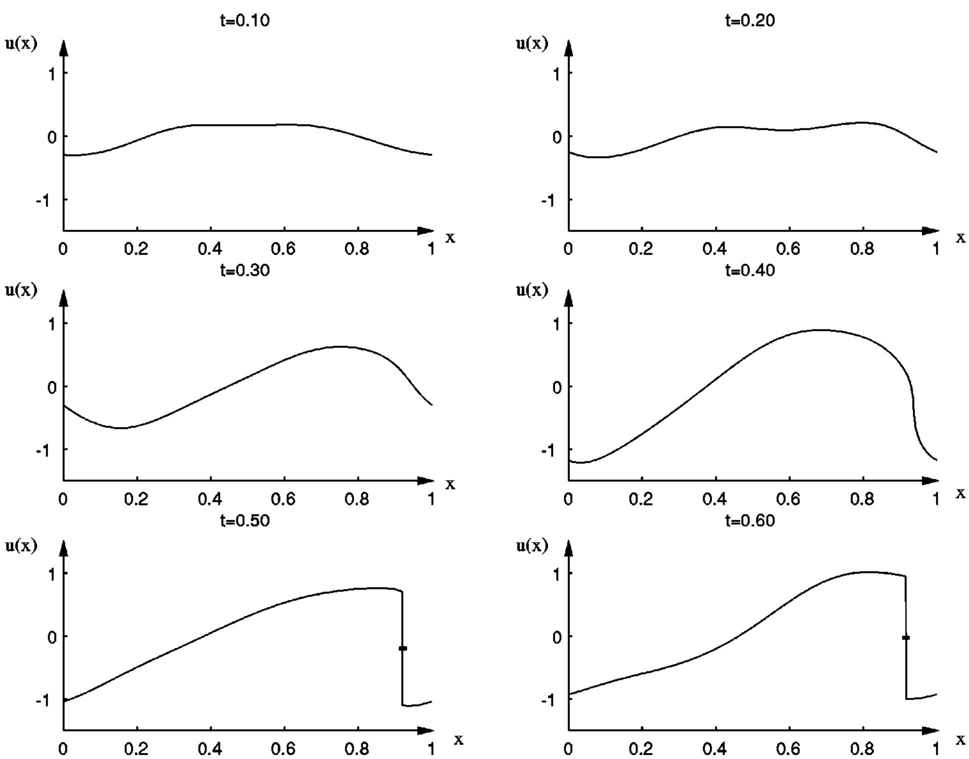

FIG. 3. Different time snapshots of gradient configuration within system size, i.e., $-\partial_x h$. The time scale for shock creation is demonstrated for $\sigma\sim L$. The solid points show the jumps in the height gradient [93].

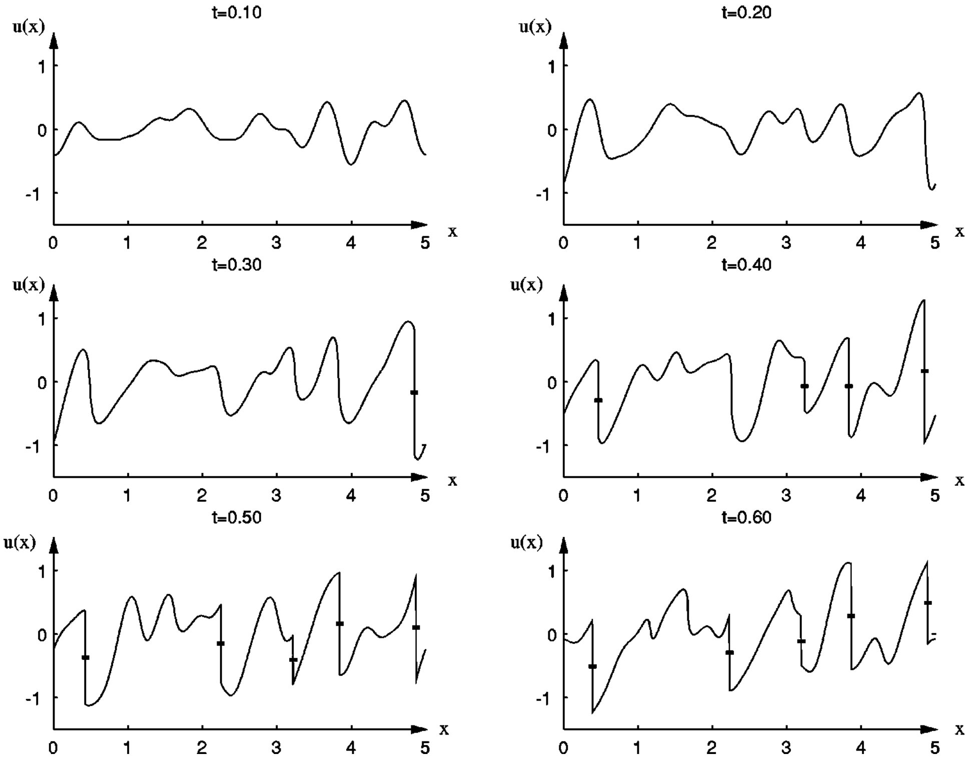

FIG. 4. Different time snapshots of gradient configuration within system size, i.e., $-\partial_x h$. The time scale for shock creation is demonstrated for $\sigma < L$. The solid points show the jumps in the height gradient [93].

$\nu \to 0$. Of course in the stationary state the sharp valleys are fully developed and one should also take into account the diffusion term in the PDF equation. The complicated term involved with the singularities can be overcome by using the method proposed in [73]. Let us define

$$G(u,\tilde{h},t) = \lim_{\nu \to 0} \nu \langle u_{xx} | u,\tilde{h} \rangle P(u,\tilde{h},t)$$
$$= \lim_{\nu \to 0} \nu \langle u_{xx}(x,t)\, \delta(\tilde{h} - \tilde{h}(x,t))\, \delta(u - u(x,t)) \rangle, \tag{46}$$

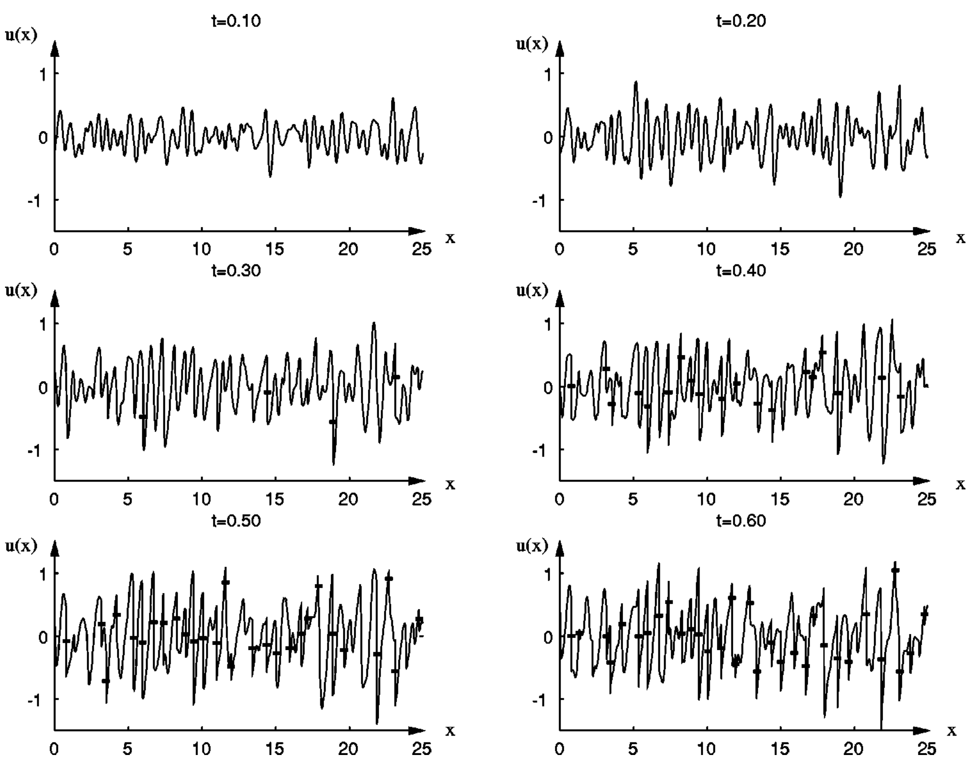

FIG. 5. Different time snapshots of gradient configuration within system size, i.e., $-\partial_x h$. The time scale for shock creation is demonstrated for $\sigma \ll L$. The solid points show the jumps in the height gradient [93].

where in the last step in Eq. (46) we have used the definition of the joint PDF $P(u,\tilde{h},t)$. Assuming spatial ergodicity, the average of the dissipative term can be expressed as

$$\nu\langle u_{xx}|u,\tilde{h}\rangle P(u,\tilde{h},t)$$
$$=\nu\Big\langle u_{xx}(x,t)\,\delta(u-u(x,t))\,\delta(h-\tilde{h}(x,t))$$
$$=\nu\lim_{L\to\infty}\frac{N}{L}\frac{1}{N}\int_{-L/2}^{L/2}dx\,u_{xx}(x,t)\,\delta(u-u(x,t))$$
$$\times\delta(\tilde{h}-\tilde{h}(x,t)). \tag{47}$$

It is well known that the u field, which satisfies Burgers equation, gives rise to discontinuous or shock solutions in the limit $\nu\to 0$. Consequently, for finite σ the shock solutions are manifested in height field as a set of sharp valleys at the positions where the shocks are located, where they are continuously connected by some hill configurations (Fig. 1). It is noted that u_{xx} is zero at the positions where no sharp valley exists. Therefore, in the limit $\nu\to 0$, only small intervals around the sharp valleys will contribute to the integral in Eq. (47). Within these intervals, boundary layer analysis can be used to obtain an accurate approximation of $u(x,t)$, $\tilde{h}(x,t)$. Generally, boundary layer analysis deals with the problems in which perturbations are operative over very narrow regions across which the dependent variables undergo very rapid changes. These narrow regions (sharp valley layers) frequently adjoin the boundaries of the domain of interest, owing to the fact that a small parameter (ν in the present problem) multiplies the highest derivative. A powerful method for treating boundary layer problems is the method of matched asymptotic expansions. The basic idea underlying this method is that an approximate solution to a given problem is sought not as a single expansion in terms of a single scale, but as two or more separate expansions in terms of two or more scales, each of which is valid in part of the domain. The scales are chosen so that the expansion as a whole covers the whole domain of interest and the domains of validity of neighboring expansions overlap. In order to handle the rapid variations in the sharp valley layers, we define a suitable magnified or stretched scale and expand the functions in terms of it in the sharp valley regions. For this purpose, we split u and h into a sum of an inner solution near the sharp valleys and an outer solution away from the sharp valleys, and we use systematic matched asymptotics to construct a uniform approximation of u and $\tilde{h}$. For the outer solution, we look for an approximation in the form of a series in ν,

$$u=u^{\rm out}=u_0+\nu u_1+O(\nu^2),$$
$$\tilde{h}=\tilde{h}^{\rm out}=\tilde{h}_0+\nu\tilde{h}_1+O(\nu^2), \tag{48}$$

where u_0 and $\tilde{h}_0$ satisfy the Burgers and KPZ equation without the dissipation term,

$$u_{0t}+\alpha u_0u_{0x}=-f_x,$$
$$\tilde{h}_{0t}-\frac{\alpha}{2}(\partial_x\tilde{h}_0)^2=f. \tag{49}$$

In order to deal with the inner solution around the sharp valley, let $y=y(t)$ be the position of a sharp valley, define the stretched variable $z=(x-y)/\nu$, and let

$$u^{\rm in}(x,t)=v\left(\frac{x-y}{\nu}+\delta,t\right),$$
$$\tilde{h}^{\rm in}(x,t)=\tilde{h}\left(\frac{x-y}{\nu}+\delta,t\right). \tag{50}$$

The parameter δ is a perturbation of the sharp valley position and v and $\tilde{h}$ satisfy the following equations:

$$\nu v_t-\alpha(\bar{u}-\nu\eta)v_z+\alpha v v_z=v_{zz}-f_z(z,t),$$
$$\nu^2\tilde{h}_t-\nu\bar{u}\tilde{h}_z+\eta\nu^2\tilde{h}_z-\frac{\alpha}{2}(\tilde{h}_z)^2=\nu\tilde{h}_{zz}+\nu^2 f(z,t). \tag{51}$$

where $\bar{u}=(1/\alpha)(dy/dt)=(u_++u_-)/2$, ($\eta=(1/\alpha)(d\delta/dt)$, and u_+, u_- are the height gradients on the in right-hand and left-hand sides of the sharp valley in the position y (see Fig. 1). We look for a solution in the form

$$v=v_0+\nu v_1+O(\nu^2),$$
$$\tilde{h}=\tilde{h}_0+\nu\tilde{h}_1+O(\nu^2). \tag{52}$$

To leading order we get for ν_0 and $\tilde{h}_0$

$$\tilde{h}_{0z}=0,$$
$$\alpha(v_0-\bar{u})v_{0z}=v_{0zz}, \tag{53}$$

where we have assumed that the variance of $f(z,t)$ is a smooth function so that we can neglect its variation in the sharp valley region ($f_z=0$). In other words, we suppose that $\sigma\gg O(\nu)$ (i.e., $\sigma\gg$ the typical layer width). One can easily integrate Eq. (53) and find that

$$\tilde{h}_0=\text{const},$$
$$v_0=\bar{u}-\frac{s}{2}\tanh\left(\frac{\alpha s z}{4}\right),$$

in which $s=s(t)=u_+-u_-$ is the shock strength. The boundary condition for the second equation arises from the matching condition,

$$\lim_{z\to\pm\infty}v_0^{\rm in}=\lim_{x\to y}u_0^{\rm out}=\bar{u}\pm\frac{s}{2}. \tag{54}$$

Basically $\tilde{h}_0^{\rm in}(z)=C-\nu\int^z v_0^{\rm in}(z')dz'$, where C is the integration constant. Therefore, the $O(1)$ solutions of $v_0^{\rm in}$ give rise to $O(\nu)$ solutions in the $\tilde{h}_0^{\rm in}$ field and only the integration constant is the $O(1)$ part of the solution of $\tilde{h}_0^{\rm in}$. In fact, the constant is merely the height value at the sharp valley position. Of course due to the height continuity at the sharp valley position there is no boundary layer for the KPZ equation, meaning that the rapid changing term in the sharp valley layer occurs in h_{xxx} while the highest derivative in the KPZ equation involves only h_{xx}. The above analysis shows that, to $O(\nu)$, Eq. (47) can be estimated as

$$\begin{aligned}
&\lim_{\nu\to 0}\nu[\langle u_{xx}|u,\tilde{h}\rangle P(u,\tilde{h},t)]\\
&\quad=\lim_{\nu\to 0}\nu\lim_{L\to\infty}\frac{N}{L}\frac{1}{N}\sum_j\int_{\Omega_j}dx\, u_{xx}^{\rm in}\delta(u-u^{\rm in}(x,t))\\
&\quad\quad\times\delta(\tilde{h}-\tilde{h}^{\rm in}(y_j,t))\\
&\quad=\lim_{L\to\infty}\frac{N}{L}\frac{1}{N}\sum_j\int_{-\infty}^{\infty}dz\, u_{zz}^{\rm in}\delta(u-u^{\rm in}(z,t))\\
&\quad\quad\times\delta(\tilde{h}-\tilde{h}^{\rm in}(y_j,t))\\
&\quad=\lim_{L\to\infty}\frac{N}{L}\frac{1}{N}\sum_j\int_{-\infty}^{\infty}dz\, v_{0zz}^{\rm in}\delta(u-v_0)\,\delta(\tilde{h}-\tilde{h}(y_j,t)),
\end{aligned}\tag{55}$$

where Ω_j is a layer located at y_j with width $\gg O(\nu)$. Using Eq. (53) and

$$dz v_{0zz}=dv_0\frac{v_{0zz}}{v_{0z}}=\alpha dv_0(v_0-\overline{u}),\tag{56}$$

the z integral can be evaluated exactly leading to the following result:

$$\begin{aligned}
\nu\langle u_{xx}|u,\tilde{h}\rangle P(u,\tilde{h},t)=\alpha\int d\overline{u}\int_{-\infty}^{0}ds\,\rho(\overline{u},s,\tilde{h},x,t)\int_{\overline{u}+(s/2)}^{\overline{u}-(s/2)}\\
\times dv_0(v_0-\overline{u})\,\delta(u-v_0).
\end{aligned}\tag{57}$$

$\varrho(\overline{u},s,\tilde{h},x,t)$ is defined such that $\varrho(\overline{u},s,\tilde{h},t)d\overline{u}\,ds\,d\tilde{h}\,dx$ gives the average number of valleys in $[x,x+dx)$ with $\overline{u}(y,t)\in[\overline{u},\overline{u}+d\overline{u})$, $s(y,t)\in[s,s+ds)$, and $\tilde{h}(y,t)\in[\tilde{h},\tilde{h}+d\tilde{h})$, where $y\in[x,x+dx)$ is the sharp valley location. Equation (57) indicates that the relaxation term in the strong-coupling limit can be written in terms of some quantities which are defined in singularities (valleys). Indeed we characterize a sharp valley with four quantities, its location y_j, its gradients at $y_{j_{0+}}$ (i.e., u_+), $y_{j_{0-}}$ (i.e., u_-), and its height from the $\overline{h}$, i.e., $\tilde{h}_j$. Instead of u_+ and u_-, we have used the quantities $\overline{u}=(u_+ + u_-)/2$ and $s=s(t)=u_+ - u_-$. Later we will determine the time evolution equations which govern these four quantities.

Proceeding further, we note that $\varrho(\overline{u},s,\tilde{h},x,t)$ can be defined as

$$\begin{aligned}
\varrho(\overline{u},s,\tilde{h},x,t)=\Big\langle\sum_j\delta(\overline{u}-\overline{u}(y_j,t))\delta(s-s(y_j,t))\delta(\tilde{h}\\
-\tilde{h}(y_j,t))\delta(x-y_j)\Big\rangle.
\end{aligned}\tag{58}$$

Due to statistical homogeneity, the sharp valley's characteristics are independent of their location, so

$$\varrho(\overline{u},s,\tilde{h},x,t)=\rho S(\overline{u},s,\tilde{h},t),\tag{59}$$

in which $\rho=\rho(t)$ is the number density of shocks and $S(\overline{u},s,\tilde{h},t)$ is the PDF of $(\overline{u}(y_0,t),s(y_0,t),\tilde{h}(y_0,t))$ conditional on y_0 being a shock location. Hence

$$\begin{aligned}
&\lim_{\nu\to 0}\nu\langle u_{xx}|u,\tilde{h}\rangle P(u,\tilde{h},t)\\
&\quad=-\alpha\rho\int_{-\infty}^{0}ds\int_{u+(s/2)}^{u-(s/2)}d\overline{u}(u-\overline{u})S(\overline{u},s,\tilde{h},t).
\end{aligned}\tag{60}$$

Therefore, the relaxation (dissipative) contribution in Eq. (20) is written as

$$G(u,\tilde{h},t)=-\left(\alpha\rho\int_{-\infty}^{0}ds\int_{u+(s/2)}^{u-(s/2)}d\overline{u}(u-\overline{u})S(\overline{u},s,\tilde{h},t)\right)_{uu}\tag{61}$$

So Eq. (20) is rewritten in the form

$$\begin{aligned}
-P_{ut}=-\gamma P_{\tilde{h}u}-\frac{\alpha}{2}(u^2P)_{\tilde{h}u}-\alpha(uP)_{\tilde{h}}-k(0)P_{\tilde{h}\tilde{h}u}\\
+k_{xx}(0)P_{uuu}+G(u,\tilde{h},t).
\end{aligned}\tag{62}$$

It is interesting that the G term comes from the relaxation term in the KPZ equation, but its explicit expression in terms of the sharp valley's characteristics is proportional to α, which is the coefficient of the nonlinear term in the KPZ equation. This indicates that without the nonlinear term in the KPZ equation there is no finite contribution for the diffusion term in the PDF equation when $\nu\to 0$. Although this equation is exact for finite σ, we cannot solve it since the last term is not expressed in terms of $P(u,\tilde{h},t)$. Despite the existence of an unclosed G term, we can still derive interesting information about the moments using the above equation. We will study comprehensively the moments of height difference and height gradient, i.e., $\langle(h-\overline{h})^n(\partial_x h)^m\rangle$, in the next section.

It is worth mentioning that integration over $\tilde{h}$ gives an equation for the PDF of u recovering the results in [73],

$$R_t=-k_{xx}(0)R_{uu}+\left\{\rho\alpha\int_{-\infty}^{0}\int_{u+(s/2)}^{u-(s/2)}d\overline{u}(u-\overline{u})S(\overline{u},s,t)\right\}_u,\tag{63}$$

where $R(u,t)=\int P(u,\tilde{h},t)d\tilde{h}$ and $S(\overline{u},s,t)=\int S(\overline{u},s,\tilde{h},t)d\tilde{h}$ is the PDF of $(\overline{u}(y_0,t),s(y_0,t))$, conditional on the property that y_0 is the singularity position. Because of the statistical homogeneity, y_0 is a dummy variable. We finish the section by determining the relation between the density of valleys and the noise characteristics $k(0)$ and $k_{xx}(0)$, that is,

$$k_{xx}(0)=\frac{\alpha\rho}{12}\langle s^3\rangle, \tag{64}$$

indicating that the forcing variance $k_{xx}(0)$ is related to the products of the density of valleys ρ, $\langle s^3\rangle$, and α. This relation has been found in [73] and its details are given in the Appendix B.

V. THE MOMENTS OF HEIGHT FLUCTUATION IN THE STATIONARY STATE

Our goal is to investigate the scaling behavior of moments of height difference and height gradient in the stationary state. After sharp valley formation, the lateral correlations produced by the nonlinear term will grow with time. The dynamic scaling exponent z characterizes the self-similar lateral growth. However, in the stationary state the height field width saturates in the sense that lateral correlations are on average grown up to the system size. As it was explained, after the saturation the width scales as $w_0(L,t\gg L^z)\sim L^{\chi}$. Having in our disposal the exact result $\chi=\frac{1}{2}$ in one dimension [89], it will be natural to define $P(h',u,t)$ as the PDF of h', u, and t, where $h'=\tilde{h})/w_0$ and $w_0=L^{1/2}$. Obviously $P(h',u,t)$ is related to $P(\tilde{h},u,t)$ as $P(h',u,t)=w_0P(\tilde{h},u,t)$. From Eq. (62), it follows that $P(h',u,t)$ in the stationary state satisfies the following equation:

$$-\gamma L^{-1/2}P_{h'u}-\frac{\alpha}{2}L^{-1/2}(u^2P)_{h'u}-\alpha L^{-1/2}(uP)_{h'}$$
$$-k(0)L^{-1}P_{h'h'u}+k_{xx}(0)P_{uuu}+G(u,h',t)=0. \tag{65}$$

From Eq. (65) it follows that the moments of $\langle h'^n u^m\rangle$ satisfy the following equation in the stationary state:

$$\begin{aligned} g_{n,m}&+k_{xx}(0)m(m-1)(m-2)\langle h'^n u^{m-3}\rangle\\ &+mn(n-1)k(0)L^{-1}\langle h'^{n-2}u^{m-1}\rangle\\ &-mn\gamma L^{-1/2}\langle h'^{n-1}u^{m-1}\rangle\\ &-n(m-2)\frac{\alpha}{2}L^{-1/2}\langle h'^{n-1}u^{m+1}\rangle=0, \end{aligned} \tag{66}$$

where

$$g_{n,m}=\int dh'\int du\, h'^n u^m G(u,h',t). \tag{67}$$

Using Eq. (61), we can determine the explicit expression of $g_{n,m}$ in terms of the characteristics of valleys. Thus we find

$$g_{n,m}=-\alpha\rho\int dh'\int du h'^n \times u^m\left(\int_{-\infty}^{0}ds\int_{u+(s/2)}^{u-(s/2)}d\overline{u}(u-\overline{u})S(\overline{u},s,h',t)\right)_{uu}. \tag{68}$$

After integrating by parts, it converts to

$$\begin{aligned} g_{n,m}&=-\alpha\rho\int dh'\int du\, h'^n m(m-1)u^{m-2}\\ &\quad\times\left(\int_{-\infty}^{0}ds\int_{u+(s/2)}^{u-(s/2)}d\overline{u}(u-\overline{u})S(\overline{u},s,h',t)\right)\\ &=-\alpha\rho m(m-1)\int_{-\infty}^{0}ds\int dh' h'^n\int du\, u^{m-2}\\ &\quad\times\left(\int_{u+(s/2)}^{u-(s/2)}d\overline{u}(u-\overline{u})S(\overline{u},s,h',t)\right). \end{aligned} \tag{69}$$

It can be integrated over u, which leads to the following expression for $g_{n,m}$ (see Appendix B):

$$\begin{aligned} g_{n,m}=\frac{\alpha\rho}{2^m}\{&\langle h_v'^n(2\overline{u}+s)^{m-1}[(m-1)s-2\overline{u}]\rangle\\ &+\langle h_v'^n(2\overline{u}-s)^{m-1}[(m-1)s+2\overline{u}]\rangle\}, \end{aligned} \tag{70}$$

where $h_v'=(h_v-\overline{h})/w_0$ and h_v is the height of a given sharp valley. This means that the relaxation term in the strong-coupling limit can be written in terms of *only* characteristics of the valleys, i.e., $\overline{u}$, s, and h_v.

At statistical steady state $(t\to\infty,\langle\;\rangle_t=0)$ and using the scale independence of $g_{n,m}$'s (see Appendix E), we derive, to leading order and in the limit of $L\to\infty$, from Eq. (66),

$$\left\langle\left(\frac{h-\overline{h}}{w_0}\right)^n u^{m-3}\right\rangle=-\frac{g_{n,m}}{k_{xx}(0)m(m-1)(m-2)} \tag{71}$$

for $m\geqslant 3$. For instance, setting $n=0$ causes the height gradient moments to behave as [73]

$$\begin{aligned} \langle u^m\rangle=&\frac{\alpha\rho}{2^{m+3}k_{xx}(0)(m+1)(m+2)(m+3)}\\ &\times\{\langle(2\overline{u}-s)^{m+2}[(m+2)s+2\overline{u}]\rangle\\ &-\langle(2\overline{u}+s)^{m+2}[2\overline{u}-(m+2)s]\rangle\}, \end{aligned} \tag{72}$$

and for $m=3$ we find

$$\langle(h-\overline{h})^n\rangle=\frac{g_{n,3}}{6k_{xx}(0)}L^{n/2}, \tag{73}$$

where, using Eq. (70), we have $g_{n,3}=(\alpha\rho/2)\langle h'^n s^3\rangle$. The fact that g_{nm}'s, up to leading order, are scale-independent implies that Eq. (73) builds up a relation between the nth moments of height difference in terms of the second moment w_0 in a nonintermittent way. This means the nth-order moment is scaled linearly with order n. To verify the scale independence of $g_{n,m}$'s, one should look at the statistics of the sharp valley environment and the different processes involved in the sharp valley creation and annihilation, which contribute dynamically. We have comprehensively explained the related arguments in Appendix E. Equation (66) also suggests that the amplitudes of height difference and height gradient moments depend strongly on the singular structures in the theory, encoded in the functions $g_{n,m}$ [i.e., Eq. (70)]. At this stage, we find the finite-size effects on the moments of $S_{n,m}=\langle h'^n u^m\rangle$. Defining $\epsilon=1/L^{1/2}$ as a perturbative parameter, we find the structure functions $S_{n,m}=\langle h'^n u^m\rangle$ perturbatively in terms of small parameter $\epsilon=1/L^{1/2}$ as

$$S_{n,m}=\langle h'^n u^m\rangle=S^{(0)}_{n,m}+\epsilon S^{(1)}_{n,m}+\epsilon^2 S^{(2)}_{n,m}+\cdots. \tag{74}$$

Using Eqs. (66), (74), and the scale independence of g_{nm}, we get

$$S^{(0)}_{n,m}=\frac{1}{k_{xx}(0)(m+1)(m+2)(m+3)}g_{n,m+3},$$

$$S^{(1)}_{n,m}=\frac{1}{k_{xx}(0)(m+1)(m+2)(m+3)}\times\left\{\frac{\gamma n(m+3)}{k_{xx}(0)(m+3)(m+4)(m+5)}g_{n-1,m+5}+\frac{\alpha n(m+1)}{2k_{xx}(0)(m+5)(m+6)(m+7)}g_{n-1,m+7}\right\}, \tag{75}$$

$$S^{(2)}_{n,m}=\frac{1}{k_{xx}(0)(m+1)(m+2)(m+3)}\times\left(\gamma n(m+3)S^{(1)}_{n-1,m+4}+\frac{\alpha}{2}n(m+1)S^{(1)}_{n-1,m+4}-k(0)n(n-1)(m+3)S^{(0)}_{n-2,m+2}\right), \tag{76}$$

etc. For example, the moments $\langle(h-\overline{h})^n\rangle$ behave as

$$\begin{aligned}\langle(h-\overline{h})^n\rangle=L^{n/2}\Bigg\{&\frac{1}{3!k_{xx}(0)}g_{n,3}\\&+\frac{1}{L^{1/2}}\left(-\frac{\gamma n}{5!k_{xx}(0)^2}g_{n-1,5}-\frac{2\alpha n}{7!k_{xx}(0)^2}g_{n-1,7}\right)\\&+\frac{1}{L}\Bigg(\frac{k(0)n(n-1)}{5!k_{xx}(0)^2}g_{n-2,5}+\frac{\gamma^2 n(n-1)}{7!k_{xx}(0)^3}g_{n-2,7}\\&-\frac{11\alpha\gamma n(n-1)}{9!k_{xx}(0)^2}g_{n-2,9}-\frac{40\alpha^2 n(n-1)}{11!k_{xx}(0)^2}g_{n-2,11}\Bigg)\\&+O(L^{-3/2})\Bigg\}.\end{aligned} \tag{77}$$

Noting that $g_{n-1,5}$ and $g_{n-1,7}$ are not zero, we conclude that the next-to-leading-order correction for the structure functions is $O(1/L^{1/2})$. Also, Eq. (77) shows that the amplitude of the correction terms to moments $\langle(h-\overline{h})^n\rangle$ is related to the statistics of quantities which are defined on the singularities, i.e., $g_{n,m}$. Also it shows that all of the moments $\langle(h-\overline{h})^n\rangle$ (for even and odd n) exist and consequently the PDF of the $h-\overline{h}$ is not symmetric. However, using the properties of the Burgers equation, it can be shown that only even moments of u are nonzero and all the odd moments vanish, hence the PDF of u is symmetric.

Equation (72) enables us to determine the rate of surface growth at the stationary state, i.e., $\overline{h_t}$. Using the KPZ equation, it is trivial to see that

$$\lim_{t\to\infty}\gamma(t)=\overline{h_t}=\frac{\alpha}{2}\langle u^2\rangle+\nu\rho\langle s\rangle, \tag{78}$$

where we have used the fact that $\langle h_{xx}\rangle=-\langle u_x\rangle=-\rho\langle s\rangle$ [73]. In the limit $\nu\to0$, the second term vanishes and

$$\lim_{t\to\infty}\gamma(t)=\overline{h_t}=\frac{\alpha}{2}\langle u^2\rangle. \tag{79}$$

Going back to Eq. (72), the $\overline{h_t}$ is written in terms of the properties of the singularities as

$$\overline{h_t}=\frac{\alpha^2\rho}{2^4 5!k_{xx}(0)}\{\langle 80\overline{u}^2 s^3+4s^5\rangle\}. \tag{80}$$

So in the stationary state, moments $\langle\overline{u}^2 s^3\rangle$ and $\langle s^5\rangle$ determine the growth rate. In other words, for a given time in the steady state, if one gets the moments $\langle\overline{u}^2 s^3\rangle$ and $\langle s^5\rangle$, which are defined only on the valleys, he can predict the rate of surface growth. This provides a simple way to determine the $\overline{h_t}$ in the stationary state. Now we prove that $P(h',u,t\to\infty,L\to\infty)=P(h',u)$ is a positive and normalizable PDF. To show the positivity of the PDF, we note that Eq. (B1) indicates that $P(h',u)$ satisfies the following equation in the limit of $L\to\infty$:

$$k_{xx}(0)P_{uuu}=\left(\alpha\rho\int_{-\infty}^{0}ds\int_{u+(s/2)}^{u-(s/2)}d\overline{u}(u-\overline{u})S(\overline{u},s,\tilde{h},t)\right)_{uu}. \tag{81}$$

Taking advantage of the method introduced in Sec. III, one may obtain

$$P(h',u)=-\frac{\alpha\rho}{2k_{xx}(0)}\int_{-\infty}^{0}ds\int_{u+(s/2)}^{u-(s/2)}\times d\overline{u}\left(\frac{s^2}{4}-(u-\overline{u})^2\right)S_\infty(\overline{u},s,h'), \tag{82}$$

where $S_\infty(\overline{u},s,h')$ is the PDF of valleys with $\overline{u}$, s, and h'. Therefore, positivity of $S_\infty(\overline{u},s,h')$ implies that $P(h',u)\geqslant0$. To check the normalizability of the $P(h',u)$, consider Eq. (71) with $n=0$, $m=3$, leading to

$$\int dh' du\, P(h',u,t) = \frac{1}{3!k_{xx}(0)} g_{0,3}, \tag{83}$$

where the explicit form of $g_{0,3}$ can be found from Eq. (70). The result is $g_{0,3}=(\alpha\rho/2)\langle s^3\rangle$. Now combining with Eq. (B15) gives

$$\int dh' du\, P(h',u,t\rightarrow\infty, L\rightarrow\infty) = 1, \tag{84}$$

so the proof of normalizability of the stationary state PDF $P(h',u)$ is completed.

VI. CONCLUSION

We study the problem of nonequilibrium surface growth described by the forced KPZ equation in $(1+1)$ dimensions. The forcing is a white in time Gaussian noise but with a Gaussian correlation in space. Modeling a short-range correlated noise, we restrict our study to the case when the correlation length of the forcing is much smaller than the system size. In the nonstationary regime when the sharp valley structures are not yet developed, we find an exact form for the generating function of the joint fluctuations of height and height gradient. We determine the time scale of the sharp valley formation and the exact functional form of the time dependence in the height difference moments at any given order. Investigating the stationary state, we give a general expression of the mixed correlations of height and height gradient at any order, in terms of the quantities which characterize the sharp valley singular structures. Through a careful analysis of the behavior of the sharp valley environment, we derive the general finite-size corrections to the scaling of an arbitrary nth moment, i.e., $\langle (h-\overline{h})^n\rangle$, at any order. Recently, Marinari *et al.* [23] have obtained the corrections to the leading-order scaling in dimensions $D=2,3,4$, in a high-resolution simulation on the RSOS discrete model, which is believed to be in the universality class of the KPZ equation stirred with a white in time Gaussian noise and δ-correlated in space. Hence they get

$$w_n(L) \sim A_n L^{n\chi}(1+B_n L^{-\omega_n}). \tag{85}$$

Irrespective of the dimension and moment order n, they observe the same subleading exponent ω_n always very close to unity (see also [91,92]). Through our calculations, we succeed in obtaining the finite-size corrections analytically. However, we have to remark that, due to working with finite correlated forcing, a firm comparison between our results and numerical simulations is not possible. More precisely, in the present paper the limiting of $\nu\rightarrow 0$ is taken into account only when σ is finite. Still the forcing correlation length is much smaller than the system size and height correlation length. But the limiting of $\sigma\rightarrow 0$ is a singular limit in our calculations, and moreover, it is not *a priori* clear that the limits of $\nu\rightarrow 0$ and $\sigma\rightarrow 0$ commute at all. However, due to the scale independence of $g_{n,m}$'s, Eq. (77) shows the general correction terms for the nth-order moment, all having the same subleading exponent $\omega_n=\frac{1}{2}$. The amplitudes A_n and B_n in Eq. (85) are given explicitly in terms of the functions $g_{n,m}$ defined on the sharp valley singularities. The next step, left for the future, would be the calculation of $g_{n,m}$'s in terms of few known parameters, i.e., the forcing and diffusion coefficients.

Our analysis enables us to find the stochastic equations which are governed over the dynamics of quantities characterizing the sharp valley singularities too. This translates the stationary nonequilibrium dynamics of the surface in terms of the dynamics of singularities in the stationary state. When the system crosses over the time t_c, after which the first singularities are formed, it would be an important study to analyze the shape deformation of nonstationary height PDF $P(h',t)$ in time. We believe that the analysis followed in this paper is quite suitable for the zero-temperature limit in the problem of a directed polymer in the random potential with short-range correlations [88]. The same method applied to the KPZ equation in higher dimensions would definitely be one of the goals of the present work. The main message that might be encoded in the present work is the importance of the statistical properties of the geometrical singular structures for understanding the strong-coupling regime of the Kardar-Parisi-Zhang equation in higher dimensions.

ACKNOWLEDGMENTS

We would like to thank J. Bec for kindly providing us with some of his unpublished numerical results. We also thank C. Castellano for reading the manuscript and directing our attention to some of the references, and Roya Mohayaee for helpful comments. We are indebted to Eric Vanden Eijnden for very informative discussions. We thank also F. Azami, R. Asgari, B. Davoudi, M. E. Fouladvand, R. Gholestanian, G. Ketabi, M. Kohandel, M. Khorammi, A. Naji, and A. R. Rasteghar for useful discussions. A.A.M. and F.S. were partially supported by the Institute for Studies in Physics and Mathematics (IPM). J.D. acknowledges partial support by the Deutsche Forschungsgemeinschaft.

APPENDIX A: AN ALTERNATIVE METHOD FOR DETERMINING THE MOMENTS OF HEIGHT FLUCTUATION BEFORE THE FORMATION OF THE SINGULARITIES

In this appendix, we give the details of calculations of the scaling behavior of moments of height difference before the formation of singularities. We know that the generating function $Z(\mu,\lambda,t)$ satisfies the following equation when ($\nu\rightarrow 0$):

$$Z_t = i\gamma\lambda Z - \frac{i\lambda\alpha}{2} Z_{\mu\mu} - \lambda^2 k(0) Z + \frac{i\alpha\lambda}{\mu} Z_\mu + \mu^2 k_{xx}(0) Z. \tag{A1}$$

Let us write $Z(\mu,\lambda,t)$ as follows:

$$Z(\mu,\lambda,t) := [1 + A(t)\lambda^2 + C(t)\lambda^3 + F(t)\lambda^4 + G(t)\lambda^5 + J(t)\lambda^6 + M(t)\lambda^7 + B(t)\lambda\mu^2 + D(t)\lambda^2\mu^2 + E(t)\lambda\mu^4 + H(t)\lambda^3\mu^2 + K(t)\lambda^2\mu^4 + L(t)\lambda^4\mu^2 + N(t)\lambda^5\mu^2 + P(t)\lambda^3\mu^4 + Q(t)\lambda\mu^6] \times \exp([-\lambda^2 k(0) + \mu^2 k_{xx}(0)]t). \tag{A2}$$

Now expanding $Z(\mu,\lambda,t)$ as a series of μ,λ and substituting it in Eq. (A1), we equate the terms in different orders of μ,λ ending with some coupled differential equations governed over the coefficients introduced in the definition of Z in Eq. (A1). So we have

$$\frac{\partial}{\partial t} A(t) = i\alpha B(t), \tag{A3}$$

$$\frac{\partial}{\partial t} B(t) = -2i\alpha k_{xx}(0)^2 t^2, \tag{A4}$$

$$\frac{\partial}{\partial t} C(t) = i\alpha D(t), \tag{A5}$$

$$\frac{\partial}{\partial t} D(t) = -4i\alpha k_{xx}(0) t B(t) - 2i\alpha E(t), \tag{A6}$$

$$\frac{\partial}{\partial t} E(t) = 0, \tag{A7}$$

$$\frac{\partial}{\partial t} F(t) = i\alpha H(t), \tag{A8}$$

$$\frac{\partial}{\partial t} H(t) = -4i\alpha k_{xx}(0) t D(t) - 2i\alpha K(t) - 2i\alpha k_{xx}(0)^2 t^2 A(t), \tag{A9}$$

$$\frac{\partial}{\partial t} K(t) = -2i\alpha k_{xx}(0)^2 t^2 B(t) - 8i\alpha k_{xx}(0) t E(t) - 9i\alpha Q(t), \tag{A10}$$

$$\frac{\partial}{\partial t} Q(t) = 0. \tag{A11}$$

By solving these differential equations with the initial conditions that $A(t),B(t),C(t),D(t),E(t),F(t),H(t),K(t),Q(t)$ are zero at $t=0$, we find

$$A(t) = \tfrac{1}{6}\alpha^2 k_{xx}(0)^2 t^4, \tag{A12}$$

$$B(t) = -\tfrac{2}{3} i\alpha k_{xx}(0)^2 t^3, \tag{A13}$$

$$C(t) = \tfrac{4}{45} i\alpha^3 k_{xx}(0)^3 t^6, \tag{A14}$$

$$D(t) = \tfrac{8}{15}\alpha^2 k_{xx}(0)^3 t^5, \tag{A15}$$

$$E(t) = 0, \tag{A16}$$

$$F(t) = -\tfrac{101}{2520}\alpha^4 k_{xx}^4 t^8, \tag{A17}$$

$$H(t) = \tfrac{101}{315} i\alpha^3 k_{xx}^4 t^7, \tag{A18}$$

$$K(t) = -\tfrac{2}{9}\alpha^2 k_{xx}(0)^4 t^6, \tag{A19}$$

$$Q(t) = 0. \tag{A20}$$

By replacing these expressions in Eq. (A2), we find $Z(\mu,\lambda,t)$ as a function of μ,λ,t explicitly without any unknown terms or expressions. Now if we expand the original form of the generating function $Z(\mu,\lambda,t)$ as a series in μ,λ, we find

$$\begin{aligned} Z(\mu,\lambda,t) &= \langle \exp\{-i\lambda[(h-\overline{h})] - i\mu[\partial_x(h-\overline{h})]\}\rangle \\ &= -\tfrac{1}{720}u^6\mu^6 - \tfrac{1}{120}(h-\overline{h})u^5\mu^5\lambda - \tfrac{1}{48}(h-\overline{h})^2u^4\mu^4\lambda^2 - \tfrac{1}{36}(h-\overline{h})^3u^3\mu^3\lambda^3 - \tfrac{1}{48}(h-\overline{h})^4u^2\mu^2\lambda^4 \\ &\quad - \tfrac{1}{120}(h-\overline{h})^5u\mu\lambda^5 - \tfrac{1}{720}(h-\overline{h})^6\lambda^6 - \tfrac{1}{120}iu^5\mu^5 - \tfrac{1}{24}i(h-\overline{h})u^4\mu^4\lambda - \tfrac{1}{12}i(h-\overline{h})^2u^3\mu^3\lambda^2 - \tfrac{1}{12}i(h-\overline{h})^3u^2\mu^2\lambda^3 \\ &\quad - \tfrac{1}{24}i(h-\overline{h})^4u\mu\lambda^4 - \tfrac{1}{120}i(h-\overline{h})^5\lambda^5 + \tfrac{1}{24}u^4\mu^4 + \tfrac{1}{6}(h-\overline{h})u^3\mu^3\lambda + \tfrac{1}{4}(h-\overline{h})^2u^2\mu^2\lambda^2 + \tfrac{1}{6}(h-\overline{h})^3u\mu\lambda^3 \\ &\quad + \tfrac{1}{24}(h-\overline{h})^4\lambda^4 + \tfrac{1}{6}iu^3\mu^3 + \tfrac{1}{2}i(h-\overline{h})u^2\mu^2\lambda + \tfrac{1}{2}i(h-\overline{h})^2u\mu\lambda^2 + \tfrac{1}{6}i(h-\overline{h})^3\lambda^3 - \tfrac{1}{2}u^2\mu^2 - (h-\overline{h})u\mu\lambda \\ &\quad - \tfrac{1}{2}(h-\overline{h})^2\lambda^2 - iu\mu - i(h-\overline{h})\lambda + 1. \end{aligned} \tag{A21}$$

Equating the coefficients of Eqs. (A2) and (A21) proportional to the same powers in μ and λ and replacing the expressions of $A(t),B(t),C(t),D(t),E(t),F(t),H(t),K(t),Q(t)$, we get the same expressions as given before, i.e.,

$$\langle (h-\overline{h})^2\rangle = -\tfrac{1}{3}t[k_{xx}(0)^2\alpha^2t^3 - 6k(0)], \tag{A22}$$

$$\langle (h-\overline{h})^3\rangle = -\tfrac{24}{45}k_{xx}(0)^3\alpha^3t^6, \tag{A23}$$

$$\langle (h-\overline{h})^4\rangle = -\tfrac{101}{105}k_{xx}(0)^4\alpha^4 t^8 - 4t^5 k_{xx}(0)^2\alpha^2 k(0) + 12t^2 k(0)^2. \tag{A24}$$

APPENDIX B: PROOF OF THE RELATION BETWEEN ρ AND $\langle S^3\rangle$

We consider the statistical steady state, i.e., $R_t=0$, so that Eq. (63) can be written as follows:

$$R_{uu}=\frac{1}{k_{xx}(0)}G(u,t) = \frac{\alpha\rho}{k_{xx}(0)}\left(\int_{-\infty}^{0}ds\int_{u+(s/2)}^{u-(s/2)}d\overline{u}(u-\overline{u})S(\overline{u},s,t)\right)_u . \tag{B1}$$

We integrate Eq. (B1) with respect to u and find

$$R_u=\frac{\alpha\rho}{k_{xx}(0)}\int_{-\infty}^{0}ds\int_{u+(s/2)}^{u-(s/2)}d\overline{u}(u-\overline{u})S(\overline{u},s,t). \tag{B2}$$

At the large time limit ($t\to\infty$), we denote R and S as R_∞ and $S_\infty(\overline{u},s)$. Therefore,

$$R_\infty=\frac{\alpha\rho}{k_{xx}(0)}\int_{-\infty}^{u}du\int_{-\infty}^{0}ds\int_{u+(s/2)}^{u-(s/2)}d\overline{u}(u-\overline{u})S_\infty(\overline{u},s). \tag{B3}$$

To determine R_∞, we define the function $K(u)$ as follows:

$$K(u)=\frac{\alpha\rho}{k_{xx}(0)}\int_{-\infty}^{0}ds\int_{u+(s/2)}^{u-(s/2)}d\overline{u}\frac{(u-\overline{u})^2}{2}S_\infty(\overline{u},s). \tag{B4}$$

Differentiating the above equation with respect to u gives us

$$\frac{d}{du}K(u)=\frac{\alpha\rho}{k_{xx}(0)}\int_{-\infty}^{0}ds\int_{u+(s/2)}^{u-(s/2)}d\overline{u}(u-\overline{u})S_\infty(\overline{u},s) + \frac{\alpha\rho}{k_{xx}(0)}\int_{-\infty}^{0}ds\frac{s^2}{8}S_\infty\left(u-\frac{s}{2},s\right) - \frac{\alpha\rho}{k_{xx}(0)}\int_{-\infty}^{0}ds\frac{s^2}{8}S_\infty\left(u+\frac{s}{2},s\right). \tag{B5}$$

Now we integrate Eq. (B5) over u from $-\infty$ to u and find

$$\int_{-\infty}^{u}du\frac{d}{du}K(u) = \frac{\alpha\rho}{k_{xx}(0)}\int_{-\infty}^{u}du\int_{-\infty}^{0}ds\int_{u+(s/2)}^{u-(s/2)}d\overline{u}(u-\overline{u}) + S_\infty(\overline{u},s)\frac{\alpha\rho}{2k_{xx}(0)}\int_{-\infty}^{u}du\int_{-\infty}^{0}ds\frac{s^2}{4}S_\infty\left(u-\frac{s}{2},s\right) - \frac{\alpha\rho}{2k_{xx}(0)}\int_{-\infty}^{u}du\int_{-\infty}^{0}ds\frac{s^2}{4}S_\infty\left(u+\frac{s}{2},s\right). \tag{B6}$$

Then we will find

$$K(u)-K(-\infty)=R_\infty(u)+\frac{\alpha\rho}{2k_{xx}(0)}\int_{-\infty}^{0}ds\int_{-\infty}^{u-(s/2)}d\overline{u}\frac{s^2}{4}S_\infty(\overline{u},s)-\frac{\alpha\rho}{2k_{xx}(0)}\int_{-\infty}^{0}ds\int_{-\infty}^{u+(s/2)}d\overline{u}\frac{s^2}{4}S_\infty(\overline{u},s). \tag{B7}$$

According to the definition of $K(u)$, we see that $K(-\infty)\to 0$ (the shock probability density function goes to zero in this limit) and therefore we find the following relation between $K(u)$ and $R_\infty(u)$:

$$K(u)=R_\infty(u)+\frac{\alpha\rho}{2k_{xx}(0)}\int_{-\infty}^{0}ds\int_{u+(s/2)}^{u-(s/2)}d\overline{u}\frac{s^2}{4}S_\infty(\overline{u},s). \tag{B8}$$

Using Eqs. (B4) and (B8), we find an explicit relation between the R_∞ and $S_\infty(\overline{u},s)$ as follows:

$$R_\infty(u)=-\frac{\alpha\rho}{2k_{xx}(0)}\int_{-\infty}^{0}ds\int_{u+(s/2)}^{u-(s/2)}d\overline{u}\left(\frac{s^2}{4}-(u-\overline{u})^2\right)\times S_\infty(\overline{u},s). \tag{B9}$$

Assuming $S_\infty(\overline{u},s)\geqslant 0$, it becomes evident that the above integral would give a realizable portability density for height gradient, that is, $R_\infty\geqslant 0$. For finite σ, Eq. (B9) gives us the PDF of height gradient in the KPZ equation in the strong-coupling limit. The function $R_\infty(u)$ enables us to determine the relation between the valleys density ρ and $k_{xx}(0)$. We would integrate over u from R_∞, so we define another function $K_1(u)$ such that

$$K_1(u)=-\frac{\alpha\rho}{2k_{xx}(0)}\int_{-\infty}^{0}ds\int_{u+(s/2)}^{u-(s/2)}d\overline{u}\left(\frac{s^2}{4}u-\frac{(u-\overline{u})^3}{3}\right)\times S_\infty(\overline{u},s), \tag{B10}$$

where differentiating $K_1(u)$ with respect to u gives

$$\frac{d}{du}K_1(u)=-\frac{\alpha\rho}{2k_{xx}(0)}\int_{-\infty}^{0}ds\int_{u+(s/2)}^{u-(s/2)}d\overline{u}\left(\frac{s^2}{4}-(u-\overline{u})^2\right)S_\infty(\overline{u},s) - \frac{\alpha\rho}{2k_{xx}(0)}\int_{-\infty}^{0}ds\left(\frac{s^2}{4}u-\frac{s^3}{24}\right)S_\infty\left(u-\frac{s}{2},s\right) + \frac{\alpha\rho}{2k_{xx}(0)}\int_{-\infty}^{0}ds\left(\frac{s^2}{4}u+\frac{s^3}{24}\right)S_\infty\left(u+\frac{s}{2},s\right). \tag{B11}$$

Now integrating the above equation over u from $-\infty$ to $+\infty$ gives

$$K_1(+\infty)-K_1(-\infty)=\int_{-\infty}^{+\infty}du\,R_\infty(u)-\frac{\alpha\rho}{2k_{xx}(0)}\int_{-\infty}^{+\infty}\times du\int_{-\infty}^{0}ds\left[\frac{s^2}{4}\left(u+\frac{s}{2}\right)-\frac{s^3}{24}\right]S_\infty(u,s)+\frac{\alpha\rho}{2k_{xx}(0)}\int_{-\infty}^{+\infty}du\int_{-\infty}^{0}\times ds\left[\frac{s^2}{4}\left(u-\frac{s}{2}\right)+\frac{s^3}{24}\right]S_\infty(u,s). \tag{B12}$$

Using the fact that $K_1(+\infty)=K_1(-\infty)=0$, we obtain

$$\int_{-\infty}^{+\infty}du\,R_\infty(u)=-\frac{\alpha\rho}{2k_{xx}(0)}\int_{-\infty}^{+\infty}du\int_{-\infty}^{0}ds\left(\frac{s^2}{4}u-\frac{s^3}{12}\right)\times S_\infty(u,s)+\frac{\alpha\rho}{2k_{xx}(0)}\int_{-\infty}^{+\infty}du\int_{-\infty}^{0}\times ds\left(\frac{s^2}{4}u+\frac{s^3}{12}\right)S_\infty(u,s), \tag{B13}$$

in which the sum of the terms on the right-hand side gives

$$\int_{-\infty}^{+\infty}du\,R_\infty(u)=\frac{\alpha\rho}{12k_{xx}(0)}\langle s^3\rangle. \tag{B14}$$

Thus from the requirement that R_∞ be normalized to unity, we get

$$k_{xx}(0)=\frac{\alpha\rho}{12}\langle s^3\rangle. \tag{B15}$$

APPENDIX C: DERIVATION OF THE FINITE CONTRIBUTION OF THE RELAXATION TERM IN THE STATIONARY STATE

In this appendix, we give the details of calculations of $g_{n,m}$ in Eq. (70). To compute g_{nm}, we introduce

$$K(u)=-m(m-1)\alpha\rho\int_{-\infty}^{0}ds\int_{-\infty}^{\infty}dh'h'^n\int_{u-(s/2)}^{u+(s/2)}\times d\bar{u}\left\{\frac{u^{m-1}}{m-1}\bar{u}-\frac{u^m}{m}\right\}S(\bar{u},s,h',t). \tag{C1}$$

By differentiating $K(u)$ and integrating in the whole range of u, we have

$$\int_{-\infty}^{\infty}\frac{dK(u)}{du}du=g_{nm}-m(m-1)\alpha\rho\int_{-\infty}^{0}ds\int_{-\infty}^{\infty}dh'h'^n\int_{-\infty}^{\infty}\times du\left\{\frac{u^m}{m}-\frac{u^{m-1}}{m-1}\left(u-\frac{s}{2}\right)\right\}S\left(u-\frac{s}{2},s,h',t\right)-m(m-1)\alpha\rho\int_{-\infty}^{0}ds\int_{-\infty}^{\infty}dh'h'^n\int_{-\infty}^{\infty}\times du\left\{\frac{u^m}{m}-\frac{u^{m-1}}{m-1}\left(u+\frac{s}{2}\right)\right\}S\left(u+\frac{s}{2},s,h',t\right). \tag{C2}$$

Since $K(+\infty)=K(-\infty)=0$, the left-hand side vanishes, therefore

$$g_{nm}=m(m-1)\alpha\rho\int_{-\infty}^{0}ds\int_{-\infty}^{\infty}dh'h'^n\int_{-\infty}^{\infty}\times d\bar{u}\left(\frac{\left(\bar{u}+\frac{s}{2}\right)^m}{m}-\frac{\left(\bar{u}+\frac{s}{2}\right)^{m-1}}{m-1}\bar{u}\right)S(\bar{u},s,h',t)-m(m-1)\alpha\rho\int_{-\infty}^{0}ds\int_{-\infty}^{\infty}dh'h'^n\int_{-\infty}^{\infty}\times d\bar{u}\left(\frac{\left(\bar{u}-\frac{s}{2}\right)^m}{m}-\frac{\left(\bar{u}-\frac{s}{2}\right)^{m-1}}{m-1}\bar{u}\right)S(\bar{u},s,h',t)=\frac{\rho}{2^m}\int_{-\infty}^{0}ds\int_{-\infty}^{\infty}d\bar{u}\int_{-\infty}^{\infty}dh'h'^n\{(2\bar{u}+s)^{m-1}\times[(m-1)s-2\bar{u}]+(2\bar{u}-s)^{m-1}\times[(m-1)s+2\bar{u}]\}S(\bar{u},s,h',t), \tag{C3}$$

which finally leads to Eq. (70).

APPENDIX D: DYNAMICS OF QUANTITIES WHICH ARE DEFINED ON THE SHARP VALLEYS

In this appendix, we determine the equation of motion for $\bar{u}(y_j,t),s(y_j,t),\tilde{h}(y_j,t)$ along the sharp valley, which is located at position y_j at time t. Using the KPZ equation and its differentiation by x around the sharp valley at position y_j, one can find a set of equations for $h_+(y_j,t)=\lim_{\epsilon-0^+}h(y_j+\epsilon,t)$, $h_-(y_j,t)=\lim_{\epsilon\to 0^+}h(y_j-\epsilon,t)$, $u_+(y_j,t)=\lim_{\epsilon\to 0^+}u(y_j+\epsilon,t)$, and $u_-(y_j,t)=\lim_{\epsilon\to 0^+}u(y_j-\epsilon,t)$ as follows:

$$h_{+t}(y_j,t)=\frac{\alpha}{2}u_+^2(y_j,t)+f(y_j,t), \tag{D1}$$

$$h_{-t}(y_j,t)=\frac{\alpha}{2}u_-^2(y_j,t)+f(y_j,t), \tag{D2}$$

$$u_{+t}(y_j,t)=-\alpha u_+(y_j,t)u_{+x}(y_j,t)-f_x(y_j,t), \tag{D3}$$

$$u_{-t}(y_j,t)=-\alpha u_-(y_j,t)u_{-x}(y_j,t)-f_x(y_j,t). \quad \text{(D4)}$$

To determine the $d/dt\{\overline{u},s,\tilde{h}\}$ we use the following identity [73,84]:

$$\begin{aligned}\frac{d}{dt}u_+(y_j,t)&=\frac{dy_j}{dt}u_{+x}(y_j,t)+u_{+t}(y_j,t)\\&=\alpha\overline{u}(y_j,t)u_{+x}(y_j,t)\\&\quad-\alpha u_{+x}(y_j,t)u_+(y_j,t)-f_x(y_j,t)\\&=-\frac{\alpha}{2}s(y_j,t)u_{+x}(y_j,t)-f_x(y_j,t),\end{aligned} \quad \text{(D5)}$$

where $\overline{u}=(1/\alpha)(dy_j/dt)$. Similarly,

$$\frac{d}{dt}\overline{u}(y_j,t)=\frac{\alpha}{2}s(y_j,t)u_{-x}(y_j,t)-f_x(y_j,t). \quad \text{(D6)}$$

These equations can be rewritten as

$$\begin{aligned}\frac{d}{dt}\overline{u}(y_j,t)&=-\frac{\alpha}{4}s(u_{+x}-u_{-x})-f_x,\\ \frac{d}{dt}s(y_j,t)&=-\frac{\alpha}{2}s(u_{+x}+u_{-x}),\end{aligned} \quad \text{(D7)}$$

where we will give the equations for $\overline{u}$ and s. Since $u=-h_x$, we write the above equations in terms of the curvature of the surface on the right and left sides of the sharp valley at position y_j as

$$\begin{aligned}\frac{d}{dt}\overline{u}(y_j,t)&=\frac{\alpha}{4}s(h_{+xx}-h_{-xx})-f_x,\\ \frac{d}{dt}s(y_j,t)&=\frac{\alpha}{2}s(h_{+xx}+h_{-xx}).\end{aligned} \quad \text{(D8)}$$

To determine the time evolution of $\tilde{h}=h-\overline{h}$, we use the KPZ equation by which one can easily show that $h_+(y_j,t)$ and $h_-(y_j,t)$ satisfy

$$\begin{aligned}\frac{d}{dt}h_+(y_j,t)&=\frac{dy_j}{dt}h_{+x}(y_j,t)+h_{+t}(y_j,t),\\ \frac{d}{dt}h_-(y_j,t)&=\frac{dy_j}{dt}h_{-x}(y_j,t)+h_{-t}(y_j,t).\end{aligned} \quad \text{(D9)}$$

By definition, we have $(d/dt)h(y_j,t)=\frac{1}{2}[(d/dt)h_+ +(d/dt)h_-]$, so using the equation for h_+ and h_-,

$$\frac{d}{dt}h(y_j,t)=-\frac{\alpha}{8}(4\overline{u}^2-s^2)+f \quad \text{(D10)}$$

and

$$\frac{d}{dt}\tilde{h}(y_j,t)=-\frac{\alpha}{8}(4\overline{u}^2-s^2)+f-\gamma, \quad \text{(D11)}$$

where $\tilde{h}(y_j,t)=h(y_j,t)-\overline{h}$ and $\overline{h}_t=\gamma$.

Therefore, in summary we have the following set of equations for a given sharp valley in the KPZ problem in the limit $\nu\to 0$:

$$\begin{aligned}\frac{dy_j}{dt}&=\alpha\overline{u},\\ \frac{d}{dt}\overline{u}(y_j,t)&=\frac{\alpha}{4}s(h_{+xx}-h_{-xx})-f_x,\\ \frac{d}{dt}s(y_j,t)&=\frac{\alpha}{2}s(h_{+xx}+h_{-xx}),\\ \frac{d}{dt}\tilde{h}(y_j,t)&=-\frac{\alpha}{8}(4\overline{u}^2-s^2)+f-\gamma.\end{aligned} \quad \text{(D12)}$$

APPENDIX E: STATISTICS FOR THE ENVIRONMENTS OF THE SINGULARITIES

In this appendix, we derive the PDF of quantities which characterize the sharp valleys. As is depicted in Fig. 2 and formerly described, the evolution of the surface after the formation of singularities is determined by the dynamics of the sharp valleys and their statistical properties. In a more quantitative sense, one should attempt to characterize the time evolution of h_{v_j}, $\overline{u}$, and s consequently. We show therefore that to leading order of the expansion in terms of system size the $g_{n,m}$'s do not depend on the scale L. Doing so, we reach such a level of describing the dynamics of the surface growth by which one may also trace the dynamics of the singularity environments. That creates a logical way to construct the pathway toward examining the statistical properties of singularity functions $g_{n,m}$. The importance of such an analysis became clear in Sec. V, in which the determination of the finite-size corrections to scaling in $S_{n,m}=\langle h'^n u^m\rangle$ was shown to be dependent on the lack of scale dependences in the singularity functions $g_{n,m}$.

Let us turn to a study of the statistics for the environment of the singularities in the KPZ equation. Define $\xi(x,t)=-h_{xx}(x,t)$ and let $W(h_u,\overline{u},s,\xi_+,\xi_-,x,t)$ be the PDF of

$$\begin{aligned}h_v(x,y_j,t)&=\tfrac{1}{2}\left[h\left(y_j+\frac{x}{2}\right)+h\left(y_j-\frac{x}{2}\right)\right],\\ \overline{u}(x,y_j,t)&=\tfrac{1}{2}\left[u\left(y_j+\frac{x}{2}\right)+u\left(y_j-\frac{x}{2}\right)\right],\\ s(x,y_j,t)&=u\left(y_j+\frac{x}{2}\right)-u\left(y_j-\frac{x}{2}\right),\\ \xi_\pm(x,y_j,t)&=-h_{x_\pm x_\pm}(y_j\pm x_\pm,t),\end{aligned}$$

conditional on y_j being a singularity position. In this section, we will find the master equation governing the evolution of $W(h_v,\overline{u},s,\xi_+,\xi_-,x,t)$ in the limit of $\nu\to 0$. Starting from the dynamical equation

$$h_t(z+x_\pm)-\frac{\alpha}{2}h_{x_\pm}^2(z+x_\pm)=f(z+x_\pm), \tag{E1}$$

$$u_t(z+x_\pm)+\alpha u u_{x_\pm}(z+x_\pm)=-f_{x_\pm}(z+x_\pm), \tag{E2}$$

$$\xi_t(z+x_\pm)+\alpha u(z+x_\pm)\xi_{x_\pm}(z+x_\pm)+\alpha^2\xi^2(z+x_\pm)$$
$$=-f_{x_\pm x_\pm}(z+x_\pm), \tag{E3}$$

we define

$$\theta(\lambda_+,\lambda_-,\mu_+,\mu_-,\eta_+,\eta_-,x_+,x_-,z,t)$$
$$=\exp[-i\lambda_+ h(z+x_+)-i\lambda_- h(z+x_-)-i\mu_+ u(z+x_+)$$
$$-i\mu_- u(z+x_-)-i\eta_+\xi(z+x_+)-i\eta_-\xi(z+x_-)] \tag{E4}$$

and

$$\Theta=\sum_j \theta\delta(z-y_j), \tag{E5}$$

then

$$\rho W(h_+,h_-,u_+,u_-,\xi_+,\xi_- x,t)$$
$$=\int\frac{d\lambda_+ d\lambda_- d\mu_+ d\mu_- d\eta_+ d\eta_-}{(2\pi)^6}$$
$$\times e^{-i\lambda_+ h_+ - i\lambda_- h_- - i\mu_+ u_+ - i\mu_- u_- - i\eta_+\xi_+ - i\eta_-\xi_-}\langle\Theta\rangle. \tag{E6}$$

Using equations (E1), (E2), (E3) we now derive equations for $\langle\Theta\rangle$ and W,

$$\langle\Theta\rangle_t=-i\lambda_+\left\langle\left(\frac{\alpha}{2}u_+^2+f_+\right)\sum_j \theta\delta(z-y_j)\right\rangle-i\lambda_-\left\langle\left(\frac{\alpha}{2}u_-^2+f_-\right)\sum_j \theta\delta(z-y_j)\right\rangle$$
$$-i\mu_+\left\langle(-\alpha u_+ u_{x_+}+f_{x_+})\sum_j \theta\delta(z-y_j)\right\rangle-i\mu_-\left\langle(-\alpha u_- u_{x_-}+f_{x_-})\sum_j \theta\delta(z-y_j)\right\rangle$$
$$-i\eta_+\left\langle(-\alpha\xi_+^2-\alpha u_+\xi_{x_+}+f_{x_+x_+})\sum_j \theta\delta(z-y_j)\right\rangle-i\eta_-\left\langle(-\alpha\xi_-^2-\alpha u_-\xi_{x_-+f_{x_-x_-}})\sum_j \theta\delta(z-y_j)\right\rangle$$
$$+\left\langle\sum_j[-\alpha\overline{u}(y_j,t)]\delta^1(z-y_j)\theta+\sum_k \delta(z-y_j)\delta(t-t_k)\theta\right\rangle-\left\langle\sum_l \delta(z-y_l)\delta(t-t_l)\theta\right\rangle. \tag{E7}$$

$\delta^1(z)=(d/dz)\delta(z)$, the (y_k,t_k)'s are the points of singularity creations, and the (y_l,t_l)'s are the points of singularity annihilation due to collisions. Assuming homogeneity and using the identity [73]

$$\theta\delta^1(z-y_j)=[\theta\delta(z-y_j)]_z-\theta_{x_+}\delta(z-y_j)-\theta_{x_-}\delta(z-y_j), \tag{E8}$$

it follows that

$$\langle\Theta\rangle_t=-i\lambda_+\left\langle\left(\frac{\alpha}{2}u_+^2+f_+\right)\sum_j \theta\delta(z-y_j)\right\rangle-i\lambda_-\left\langle\left(\frac{\alpha}{2}u_-^2+f_-\right)\sum_j \theta\delta(z-y_j)\right\rangle-i\mu_+\left\langle(-\alpha u_+ u_{x_+}+f_{x_+})\sum_j \theta\delta(z-y_j)\right\rangle$$
$$-i\mu_-\left\langle(-\alpha u_- u_{x_-}+f_{x_-})\sum_j \theta\delta(z-y_j)\right\rangle-i\eta_+\left\langle(-\alpha\xi_+^2-\alpha u_+\xi_{x_+}+f_{x_+x_+})\sum_j \theta\delta(z-y_j)\right\rangle$$
$$-i\eta_-\left\langle(-\alpha\xi_-^2-\alpha u_-\xi_{x_-}+f_{x_-x_-})\sum_j \overline{u}(y_j,t)(\theta_{x_+}+\theta_{x_-})\delta(z-y_j)\right\rangle+\Sigma_1-\Sigma_2, \tag{E9}$$

where Σ_1 and Σ_2 account, respectively, for singularity creation and collision events. These are given by

$$\Sigma_1(\lambda_+,\lambda_-,\mu_+,\mu_-,\eta_+,\eta_-,x_+,x_-,z,t)$$
$$=\left\langle\sum_k \theta\delta(z-y_k)\delta(t-t_k)\right\rangle, \tag{E10}$$

$$\Sigma_2(\lambda_+,\lambda_-,\mu_+,\mu_-,\eta_+,\eta_-,x_+,x_-,z,t)$$
$$=\left\langle\sum_l \theta\delta(z-y_l)\delta(t-t_l)\right\rangle. \tag{E11}$$

In regard to Novikov's theorem, we have

$$\langle f_{\pm}\theta\rangle=[-i\lambda_{\pm}k(0)-i\lambda_{\mp}k(x_{\pm}-x_{\mp})-i\mu_{\mp}k_x(x_{\pm}-x_{\mp})+i\eta_{\pm}k_{xx}(0)+i\eta_{\mp}(x_{\pm}-x_{\mp})]\langle\theta\rangle, \tag{E12}$$

$$\langle f_{x_{\pm}}\theta\rangle=[-i\lambda_{\pm}k_x(x_{\pm}-x_{\mp})-i\mu_{\pm}k_{xx}(0)+i\mu_{\mp}k_x(x_{\pm}-x_{\mp})+i\eta_{\mp}k_{xxx}(x_{\pm}-x_{\mp})]\langle\theta\rangle, \tag{E13}$$

$$\langle f_{x_{\pm}x_{\pm}}\theta\rangle=[-i\lambda_{\pm}k_{xx}(x_{\pm}-x_{\mp})-i\lambda_{\mp}k_{xx}(x_{\pm}-x_{\mp})-i\mu_{\mp}k_{xxx}(x_{\pm}-x_{\mp})+i\eta_{\pm}k_{xxxx}(0)+i\eta_{\mp}k_{xxxx}(x_{\pm}-x_{\mp})]\langle\theta\rangle. \tag{E14}$$

To average the convective terms, we use

$$i\alpha\mu_{\pm}\langle u_{\pm}\xi_{\pm}\Theta\rangle+i\alpha\eta_{\pm}\langle u_{\pm}\xi_{\pm}\Theta\rangle=-\alpha\langle u_{\pm}\Theta_{x_{\pm}}\rangle+i\alpha\lambda_{\pm}\langle u_{\pm}^2\Theta\rangle=-\alpha\langle u_{\pm}\Theta\rangle_{x_{\pm}}+\alpha\langle\xi_{\pm}\Theta\rangle+i\alpha\lambda_{\pm}\langle u_{\pm}^2\Theta\rangle=-i\alpha\langle\Theta\rangle_{x_{\pm}\mu_{\pm}}+i\alpha\langle\Theta\rangle_{\eta_{\pm}}-i\alpha\lambda_{\pm}\langle\Theta\rangle_{\mu_{\pm}\mu_{\pm}}. \tag{E15}$$

Note that

$$\theta_{x_{\pm}}=(i\lambda_{\pm}u_{\pm}-i\mu_{\pm}\xi_{x_{\pm}}-i\eta_{\pm}\xi_{x_{\pm}})\theta, \tag{E16}$$

$$\langle u_{\pm}^2\Theta\rangle=-\langle\Theta\rangle_{\mu_{\pm}\mu_{\pm}}, \tag{E17}$$

$$\langle \xi_{\pm}^2\Theta\rangle=-\langle\Theta\rangle_{\eta_{\pm}\eta_{\pm}}. \tag{E18}$$

Finally, for $\langle\Theta\rangle_t$ we find

$$\begin{aligned}\langle\Theta\rangle_t=&-\frac{i\alpha}{2}\lambda_+\langle\Theta\rangle_{\mu_+\mu_+}-\frac{i\alpha}{2}\lambda_-\langle\Theta\rangle_{\mu_-\mu_-}-\frac{i\alpha}{2}(\langle\Theta\rangle_{x_+\mu_+}+\langle\Theta\rangle_{x_-\mu_-}-\langle\Theta\rangle_{x_+\mu_-}-\langle\Theta\rangle_{x_-\mu_+})+\frac{i\alpha}{2}(\langle\Theta\rangle_{\eta_+}+\langle\Theta\rangle_{\eta_-})\\&-i\alpha\eta_+\langle\Theta\rangle_{\eta_+\eta_+}-i\alpha\eta_-\langle\Theta\rangle_{\eta_-\eta_-}-[\lambda_+^2k(0)+\lambda_-^2k(0)+2\lambda_+\lambda_-k(x_+-x_-)]\langle\Theta\rangle\\&-[\mu_+^2k_{xx}(0)+\mu_-^2k_{xx}(0)+2\mu_+\mu_-k_{xx}(x_+-x_-)]\langle\Theta\rangle-[\eta_+^2k_{xxxx}(0)+\eta_-^2k_{xxxx}(0)+2\eta_+\eta_-k(x_+-x_-)]\langle\Theta\rangle\\&-2(\lambda_+\mu_--\lambda_-\mu_+)k_x(x_+-x_-)\langle\Theta\rangle+2(\lambda_+\eta_++\lambda_-\eta_-)k_{xx}(0)\langle\Theta\rangle+2(\lambda_+\eta_-+\lambda_-\eta_+)k_{xx}(x_+-x_-)\langle\Theta\rangle\\&-2(\eta_-\mu_+-\eta_+\mu_-)k_{xxx}(x_+-x_-)\langle\Theta\rangle+\left\langle\alpha\sum_j\overline{u}(y_j,t)(\theta_{x_+}+\theta_{x_-})\delta(z-y_j)\right\rangle+\Sigma_1-\Sigma_2.\end{aligned} \tag{E19}$$

For the term involving $\overline{u}(y_j,t)$, we note that

$$u_{\pm}(y_j,t)\theta_{x_{\pm}}=[u(y_j+x_{\pm},t)\theta]_{x_{\pm}}-\xi(y_j+x_{\pm},t)\theta=i\theta_{x_{\pm}}\lambda_{\pm}-i\theta_{\mu_{\pm}}, \tag{E20}$$

$$u_{\pm}(y_j,t)\theta_{x_{\mp}}=[u(y_j+x_{\pm},t)\theta]_{x_{\mp}}=i\theta_{x_{\mp}}\lambda_{\pm}, \tag{E21}$$

thus

$$\alpha\left\langle\sum_j\overline{u}(y_j,t)(\theta_{x_+}+\theta_{x_-})\delta(z-y_j)\right\rangle=\frac{i\alpha}{2}(\langle\Theta\rangle_{x_+\lambda_+}+\langle\Theta\rangle_{x_-\lambda_+}+\langle\Theta\rangle_{x_+\lambda_-}+\langle\Theta\rangle_{x_-\lambda_-}-\langle\Theta\rangle_{\mu_+}-\langle\Theta\rangle_{\mu_-}). \tag{E22}$$

Combining the above expressions, on the subset $\lambda_+=\lambda_-=\lambda/2$, $x_+=-x_-=x/2$, $\mu_1=\mu_++\mu_-$, and $\mu_2=(\mu_+-\mu_-)/2$, $\langle\Theta\rangle$ satisfies

$$\begin{aligned}\langle\Theta\rangle_t=&-\frac{i\alpha}{4}\lambda(2\langle\Theta\rangle_{\mu_1\mu_1}+\tfrac{1}{2}\langle\Theta\rangle_{\mu_2\mu_2})-i\alpha\langle\Theta\rangle_{x\mu_2}+\frac{i\alpha}{2}(\langle\Theta\rangle_{\eta_+}+\langle\Theta\rangle_{\eta_-})-i\alpha\eta_+\langle\Theta\rangle_{\eta_+\eta_+}-i\alpha\eta_-\langle\Theta\rangle_{\eta_-\eta_-}\\&-\frac{\lambda^2}{2}[k(0)+k(x)]\langle\Theta\rangle-\left(\frac{\mu_1^2}{2}+2\mu_2^2\right)k_{xx}(0)-2\left(\frac{\mu_1^2}{4}-\mu_2^2\right)k_{xx}(x)\langle\Theta\rangle-[\eta_+^2k_{xxxx}(0)+\eta_-^2k_{xxxx}(0)+2\eta_+\eta_-k_{xxxx}(x)]\\&\times\langle\Theta\rangle+2\lambda\mu_2k_x(x)\langle\Theta\rangle+\lambda(\eta_++\eta_-)[k_{xx}(0)+k_{xx}(x)]\langle\Theta\rangle\\&+2\mu_2(\eta_++\eta_-)k_{xxx}(x)\langle\Theta\rangle+\mu_1(\eta_--\eta_+)k_{xxxx}(x)\langle\Theta\rangle+\Sigma_2-\Sigma_1.\end{aligned}$$

The Σ_1,Σ_2 are evaluated at $\lambda_+=\lambda_-=\lambda/2$, $x_+=-x_-=x/2$, $\mu_1=\mu_++\mu_-$, and $\mu_2=(\mu_+-\mu_-)/2$. Changing to the variables $(h_v,\overline{u},s,\xi_+,\xi_-)$, we obtain the following equation for W:

$$[\rho W(h_v,\overline{u},s,\xi_+,\xi_-,x,t)]_t=\frac{\alpha}{2}\overline{u}^2\rho W_{h_v}+\frac{\alpha}{8}s^2\rho W_{h_v}-\alpha s\rho W_x+\frac{\alpha}{2}\xi_+\rho W+\frac{\alpha}{2}\xi_-\rho W+\alpha\rho(\xi_+^2 W)_{\xi_+}+\alpha\rho(\xi_-^2 W)_{\xi_-}$$
$$+k(0)\rho W_{h_v h_v}+k_{xx}(0)\rho W_{\overline{uu}}+2\rho[k_{xx}(0)-k_{xx}(x)]W_{ss}+\rho k_{xxxx}(0)(W_{\xi_+\xi_+}+W_{\xi_-\xi_-})$$
$$+2\rho k_{xxxx}(x)W_{\xi_-\xi_-}-2k_x(x)\rho W_{sh_v}-2k_{xx}(0)\rho W_{h_v\xi_+}-2k_{xx}(0)\rho W_{h_v\xi_-}-2k_{xx}(x)\rho W_{h_v\xi_+}$$
$$-2k_{xx}(x)\rho W_{h_v\xi_-}-2\rho k_{xxx}(x)(W_{s\xi_-}+W_{s\xi_+})+k_{xxx}(x)\rho W_{\overline{u}\xi_-}+k_{xxx}(x)\rho W_{\overline{u}\xi_+}+\zeta_1-\zeta_2. \tag{E23}$$

The $\zeta_1(h_{v_j},\overline{u},s,\xi_+,\xi_-,x,t)$ is defined such that

$$\zeta_1(h_{v_j},\overline{u},s,\xi_+,\xi_-,x,t)dh_{v_j}ds\,d\overline{u}\,d\xi_+d\xi_-dz\,dt \tag{E24}$$

gives the average number of singularity creation points in $[z,z+dz)\times[t,t+dt)$ with

$$h_v(x,y_1,t_1)\in[h_v,h_v+dh_v),$$
$$\overline{u}(x,y_1,t_1)\in[\overline{u},\overline{u}+d\overline{u}),$$
$$s(x,y_1,t_1)\in[s,s+ds),$$
$$\xi\left(y_1+\frac{x}{2},t_1\right)\in[\xi_+,\xi_++d\xi_+),$$
$$\xi\left(y_1-\frac{x}{2},t_1\right)\in[\xi_-,\xi_-+d\xi_-),$$

conditional on $(y_1,t_1)\in([z,z+dz)\times[t,t+dt))$ being a point of singularity creation (because of the statistical homogeneity, z is a dummy variable). $\zeta_2(h_{v_j},\overline{u},s,\xi_+,\xi_-,x,t)$ is defined such that

$$\zeta_2(h_{v_j},\overline{u},s,\xi_+,\xi_-,x,t)dh_{v_j}ds\,d\overline{u}\,d\xi_+d\xi_-dz\,dt \tag{E25}$$

gives the average number of singularity collision points in $[z,z+dz)\times[t,t+dt)$ with

$$h_v(x,y_2,t_2)\in[h_v,h_v+dh_v),$$
$$\overline{u}(x,y_2,t_2)\in[\overline{u},\overline{u}+d\overline{u}),$$
$$s(x,y_2,t_2)\in[s,s+ds),$$
$$\xi\left(y_2+\frac{x}{2},t_2\right)\in[\xi_+,\xi_++d\xi_+),$$
$$\xi\left(y_2-\frac{x}{2},t_2\right)\in[\xi_-,\xi_-+d\xi_-),$$

conditional on $(y_2,t_2)\in([z,z+dz)\times[t,t+dt))$ being a point of singularity collision. Now we rescale h_v as $h_v'=h_v/L^{1/2}$, so Eq. (E23) changes to

$$L^{-1/2}\frac{\alpha}{2}\overline{u}^2\rho W'_{h_v'}+L^{-1/2}\frac{\alpha}{8}s^2\rho W'_{h_v'}-\alpha s\rho W'_x+\frac{\alpha}{2}\xi_+\rho W'+\frac{\alpha}{2}\xi_-\rho W'+\alpha\rho(\xi_+^2W')_{\xi_+}+\alpha\rho(\xi_-^2W')_{\xi_-}+L^{-1}k(0)\rho W'_{h_v'h_v'}$$
$$+k_{xx}(0)\rho W'_{\overline{uu}}+2\rho[k_{xx}(0)-k_{xx}(x)]W'_{ss}+\rho k_{xxxx}(0)(W'_{\xi_+\xi_+}+W'_{\xi_-\xi_-})+2\rho k_{xxxx}(x)W'_{\xi_-\xi_-}-2L^{-1/2}k_x(x)\rho W'_{sh_v'}$$
$$-2k_{xx}(0)L^{-1/2}\rho W'_{h_v'\xi_+}-2k_{xx}(0)L^{-1/2}\rho W'_{h_v'\xi_-}-2k_{xx}(x)L^{-1/2}\rho W'_{h_v'\xi_+}-2k_{xx}(x)L^{-1/2}\rho W'_{h_v'\xi_-}$$
$$-2\rho k_{xxx}(x)(W'_{s\xi_-}+W'_{s\xi_+})-k_{xxx}(x)\rho W'_{\overline{u}\xi_-}+k_{xxx}(x)\rho W'_{\overline{u}\xi_+}+\zeta_1'-\zeta_2'=0. \tag{E26}$$

In the limit of large L or $L\to\infty$, the leading terms are

$$-\alpha s\rho W'_x+\frac{\alpha}{2}\xi_+\rho W'+\frac{\alpha}{2}\xi_-\rho W'+\alpha\rho(\xi_+^2W')_{\xi_+}+\alpha\rho(\xi_-^2W')_{\xi_-}+k_{xx}(0)\rho W'_{\overline{uu}}+2\rho[k_{xx}(0)-k_{xx}(x)]W'_{ss}+\rho k_{xxxx}(0)$$
$$\times(W'_{\xi_+\xi_+}+W'_{\xi_-\xi_-})+2\rho k_{xxxx}(x)W'_{\xi_-\xi_-}-2\rho k_{xxx}(x)(W'_{s\xi_-}+W'_{s\xi_+})-k_{xxx}(x)\rho W'_{\overline{u}\xi_-}+k_{xxx}(x)\rho W'_{\overline{u}\xi_+}+\zeta_1'-\zeta_2'=0. \tag{E27}$$

To find the $g_{n,m}$ we multiply the above equation by $h'^n\overline{u}^m s^p$, and integrating over h', $\overline{u}$, s, ξ_+, and ξ_-, we have

$$-\alpha\rho\langle h_v'^n\overline{u}^m s^{p+1}\rangle_x+\frac{\alpha\rho}{2}\langle h_v'^n\overline{u}^m s^p\xi_+\rangle+\frac{\alpha\rho}{2}\langle h_v'^n\overline{u}^m s^p\xi_-\rangle +2p(p-1)\rho[k_{xx}(0)-k_{xx}(x)]\langle h_v'^n\overline{u}^m s^{p-2}\rangle+m(m-1)k_{xx}(0)\rho\langle h_v'^n\overline{u}^{m-2}s^p\rangle+Q_{nmp}^{(1)}-Q_{nmp}^{(2)}=0, \tag{E28}$$

where

$$Q_{nmp}^{(1)}=\int h'^n\overline{u}^m s^p\zeta_1' dh'\,d\overline{u}\,ds\,d\xi_+d\xi_-,$$

$$Q_{nmp}^{(2)}=\int h'^n\overline{u}^m s^p\zeta_2' dh'\,d\overline{u}\,ds\,d\xi_+d\xi_-.$$

Using the identities

$$\frac{\partial}{\partial x}h_v'(x,y_j,t)=-\frac{s}{4L^{1/2}}, \tag{E29}$$

$$\frac{\partial}{\partial x}s(x,y_j,t)=\tfrac{1}{2}(\xi_++\xi_-), \tag{E30}$$

$$\frac{\partial}{\partial x}\overline{u}(x,y_j,t)=\frac{s}{2}, \tag{E31}$$

we find

$$\frac{\alpha\rho}{2}\langle h_v'^n\overline{u}^m s^p(\xi_++\xi_-)\rangle =\frac{\alpha\rho}{p+1}\left\{\langle h_v'^n\overline{u}^m s^{p+1}\rangle_x+\frac{n}{4L^{1/2}}\langle h_v'^{(n-1)}S\overline{u}^m s^{p+2}\rangle +\frac{m}{2}\langle h_v'^n\overline{u}^{m-1}s^{p+2}\rangle\right\}. \tag{E32}$$

So in the limit of $L\to\infty$, we have

$$-\frac{p\alpha\rho}{p+1}\langle h_v'^n\overline{u}^m s^{p+1}\rangle_x+\frac{m\alpha\rho}{2(p+1)}\langle h_v'^n\overline{u}^{m-1}s^{p+2}\rangle +2p(p-1)\rho[k_{xx}(0)-k_{xx}(x)]\langle h_v'^n\overline{u}^m s^{p-2}\rangle +m(m-1)k_{xx}(0)\rho\langle h_v'^n\overline{u}^{m-2}s^p\rangle+Q_{nmp}^{(1)}-Q_{nmp}^{(2)}=0. \tag{E33}$$

Assuming a stationary solution for the dynamical equation (E23) governed over W and rescaling h_v as $h_v'=h_v/L^{1/2}$ in the resulting differential equation, we reach Eq. (E26). Of course the mentioned equation *is* dependent on scale L, but being interested in the limit of $L\to\infty$ results in Eq. (E27), which is free of the explicit scale-dependent terms in the leading order. However, we are faced with two very complicated terms, namely ζ_1 and ζ_2, which should be analyzed. The origin of these terms is related to processes of sharp valley creation and annihilation. We argue that these processes basically involve local interaction between nearby sharp valleys and the effects of forcing, which spatial correlation is assumed to be much less than system size, so they essentially would not carry any information about system size. In this sense, Eq. (E27) encodes the fact that the probability distribution W is a scale-invariant function of its argument $h_v'=h_v/L^{1/2}$ in the leading order. The above property is deciphered in Eq. (E33) too, but this time it is translated in terms of the scale independence of $g_{n,m}$'s.

Also, the equation for W enables us to find the time evolution of the sharp valley characteristics. For example, multiplying Eq. (E23) by h_v and integrating over all variables, we can derive the increasing rate of mean height of the singularities, and noting that

$$\frac{\partial}{\partial x}h_v(x,y_j,t)=-\frac{s}{4}, \tag{E34}$$

$$\frac{\partial}{\partial x}s(x,y_j,t)=\tfrac{1}{2}(\xi_++\xi_-), \tag{E35}$$

we get

$$\frac{d}{dt}\langle h_v\rangle(t)=-\frac{\alpha}{8}\langle 4\overline{u}^2-s^2\rangle. \tag{E36}$$

[1] A.-L. Barabasi and H. E. Stanley, *Fractal Concepts in Surface Growth* (Cambridge University Press, New York, 1995).
[2] T. Halpin-Healy and Y. C. Zhang, Phys. Rep. **254**, 218 (1995); J. Krug, Adv. Phys. **46**, 139 (1997).
[3] J. Krug and H. Spohn in *Solids Far from Equilibrium Growth, Morphology and Defects*, edited by C. Godreche (Cambridge University Press, New York, 1990).
[4] P. Meakin, *Fractals, Scaling and Growth Far from Equilibrium* (Cambridge University Press, Cambridge, 1998).
[5] Mehran Kardar, Physica A **281**, 295 (2000).
[6] S. F. Edwards and D. R. Wilkinson, Proc. R. Soc. London, Ser. A **381**, 17 (1982).
[7] M. Kardar, G. Parisi, and Y. C. Zhang, Phys. Rev. Lett. **56**, 889 (1986).
[8] F. Family and T. Vicsek, J. Phys. A **18**, L75 (1985).
[9] M. Marsilli, A. Maritan, F. Toigoend, and J. R. Banavar, Rev. Mod. Phys. **68**, 963 (1996).
[10] P. Meakin, P. Ramanlal, L. M. Sander, and R. C. Ball, Phys. Rev. A **34**, 5091 (1986).
[11] J. Krug, Phys. Rev. A **36**, 5465 (1987).
[12] U. Dekker and F. Haake, Phys. Rev. A **11**, 2043 (1975).
[13] E. Frey and V. C. Tauber, Phys. Rev. E **50**, 1024 (1994).
[14] K. J. Wiese, Phys. Rev. E **56**, 5013 (1997).
[15] F. Colaiori and M. A. Moore, e-print cond-mat/0010410.
[16] E. Katzav and M. Schwartz, e-print cond-mat/0012103.
[17] L. H. Tang and H. Leschorn, Phys. Rev. A **45**, 7162 (1992).

[18] T. Ala-Nissila, T. Hielt, J. M. Kosterlitz, and O. Venalainen, J. Stat. Phys. **72**, 207 (1993).
[19] J. M. Kim, Phys. Rev. Lett. **80**, 888 (1998).
[20] C. S. Chin and M. den Nijs, e-print cond-mat/9810083.
[21] M. Prahofer and H. Spohn, e-print cond-mat/9912264.
[22] M. K. Verma, Physica A **277**, 359 (2000).
[23] E. Marinari, A. Pagnani, and G. Parisi, J. Phys. A **33**, 8181 (2000).
[24] M. A. Moore *et al.*, Phys. Rev. Lett. **74**, 4257 (1995).
[25] T. J. Newman and H. Kallabis, e-print cond-mat/9512104.
[26] T. J. Newman and A. J. Bray, e-print cond-mat/9604071.
[27] S. Stepanow, Phys. Rev. E **55**, R4853 (1997).
[28] M. Lassig, Nucl. Phys. B **448**, 559 (1998).
[29] C. Castellano, M. Marsili, and L. Pietronero, Phys. Rev. Lett. **80**, 3525 (1998).
[30] C. Castellano, A. Gabrielli, M. Marsili, M. A. Munoz, and L. Pietronero, Phys. Rev. E **58**, R5209 (1998).
[31] C. Castellano, M. Marsili, M. A. Munoz, and L. Pietronero, e-print cond-mat/9904434.
[32] H. K. Janssen, U. C. Taeuber, and E. Frey, Eur. Phys. J. B **9**, 491 (1999).
[33] L. Golubovic and Z. G. Wang, Phys. Rev. E **49**, 2567 (1994).
[34] I. Ispolatov, P. L. Krapirsky, and S. Redner, Phys. Rev. E **52**, 2540 (1995).
[35] J. Krug and H. Spohn, Europhys. Lett. **8**, 219 (1989).
[36] H. Van Beijeren, R. Kutner, and H. Spohn, Phys. Rev. Lett. **54**, 2026 (1985).
[37] H. K. Janseen and B. Schmittmann, Z. Phys. B: Condens. Matter **63**, 517 (1986).
[38] G. Blatter, M. V. Feigelman, V. B. Geshkenbein, A. I. Larkin, and V. M. Vinokur, Rev. Mod. Phys. **66**, 1125 (1994).
[39] D. A. Huse, C. L. Henley, and D. S. Fisher, Phys. Rev. Lett. **55**, 2924 (1985).
[40] M. Kardar and Y. C. Zhang, Phys. Rev. Lett. **58**, 2087 (1987).
[41] D. S. Fisher and D. A. Huse, Phys. Rev. B **43**, 10728 (1991).
[42] R. D. Kamien, P. Le. Doussal, and D. R. Nelson, Phys. Rev. A **45**, 8727 (1992).
[43] H. C. Fogedby, A. B. Eriksson, and L. V. Mikheev, Phys. Rev. Lett. **75**, 1883 (1995).
[44] L. Balents, J-P. Bouchaud, and M. Mezard, e-print cond-mat/9601137.
[45] R. Bundschuh and M. Lassig, e-print cond-mat/9602045.
[46] G. Parisi and F. Slanina, e-print cond-mat/9712208.
[47] D. A. Gorokhov and G. Blatter, Phys. Rev. Lett. **82**, 2705 (1999).
[48] P. Delos Rios, Phys. Rev. Lett. **82**, 4236 (1999).
[49] A. Basu, Phys. Rev. E **62**, 4675 (2000).
[50] E. Brunet and B. Derrida, Phys. Rev. E **61**, 6789 (2000).
[51] E. Perlsman and S. Havlin, Phys. Rev. E **63**, 010102(R) (2001).
[52] R. Mohayaee, A. L. Stella, and C. Vander Zande, e-print cond-mat/0101091.
[53] M. V. Feigelman, Sov. Phys. JETP **52**, 555 (1980) [Zh. Eksp. Teor. Fiz. **79**, 1095 (1980)].
[54] S. F. Shandarin and Ya. B. Zeldovich, Rev. Mod. Phys. **61**, 185 (1989).
[55] M. Vergassola, B. Dubrulle, U. Frisch, and A. Noullez, Astron. Astrophys. **280**, 325 (1994).
[56] S. F. Shandarin, astro-ph/9507082.
[57] U. Frisch, J. Bec, and B. Villone, e-print cond-mat/9912110.
[58] J. P. Bouchaud, M. Mezard, and G. Parisi, Phys. Rev. E **52**, 5116 (1995).
[59] M. Mezard, e-print cond-mat/9801029.
[60] H. C. Fogedby, Phys. Rev. E **57**, 2331 (1998); Phys. Rev. Lett. **80**, 1126 (1998); Phys. Rev. E **57**, 4993 (1998); **59**, 5065 (1999); **66**, 4950 (1999).
[61] V. Yakhot and A. Chekhlov, Phys. Rev. Lett. **77**, 3118 (1996).
[62] A. Chekhlov and V. Yakhot, Phys. Rev. E **51**, R2739 (1995).
[63] A. Polyakov, Phys. Rev. E **52**, 6183 (1995).
[64] E. Balkovsky, G. Falkovich, I. Kolokolov, and V. Lebedev, JETP Lett. **61**, 1012 (1995); Phys. Rev. Lett. **78**, 1452 (1997); e-print chao-dyn/9609005.
[65] G. Falkovich and V. Lebedev, e-print chao-dyn/9708002.
[66] V. Gurarie and A. Migdal, Phys. Rev. E **54**, 4908 (1996).
[67] J. P. Bouchaud and M. Mezard, Phys. Rev. E **54**, 5116 (1996).
[68] T. Gotoh and R. H. Kraichnan, e-print chao-dyn/9803037.
[69] S. Boldyrev, Phys. Rev. E **55**, 6907 (1997).
[70] S. Boldyrev, e-print hep-th/9707255; e-print hep-th/9805100.
[71] V. Yakhot, Phys. Rev. E **57**, 1737 (1997); e-print chao-dyn/9904016; e-print chao-dyn/9909017; e-print chao-dyn/0001027.
[72] W. E. K. Khanin, A. Mazel, and Ya. G. Sinai, Phys. Rev. Lett. **78**, 1904 (1997).
[73] W. E and E. Vanden Eijnden, Phys. Rev. Lett. **83**, 2572 (1999); e-print chao-dyn/9901006; e-print chao-dyn/9904028; e-print chao-dyn/9901029.
[74] R. H. Kraichnan, Phys. Fluids **11**, 3738 (1999); e-print chao-dyn/9901023.
[75] T. Gotoh and R. H. Kraichnan, Phys. Fluids A **5**, 445 (1993); Phys. Fluids **10**, 2859 (1998).
[76] J. Bec and U. Frisch, e-print cond-mat/9906047; J. Bec, e-print nlin.CD/0103029.
[77] J. Bec, U. Frisch, and K. Khanin, e-print chao-dyn/9910001.
[78] D. Bernard and K. Gawedzki, e-print chao-dyn/9805002.
[79] Michel Bauer and Denis Bernard, e-print chao-dyn/9812018.
[80] L. Frachebourg and Ph. A. Martin, e-print cond-mat/9905056.
[81] S. N. Gurbatov, e-print chao-dyn/9912011.
[82] F. Hayot and C. Jayaprakash, e-print nlin.CD/0005050; e-print chao-dyn/9901026.
[83] U. Frisch and J. Bec, e-print nlin.CD/0012033.
[84] D. Bernard, e-print cond-mat/0007106.
[85] J. Davoudi, A. A. Masoudi, M. R. Rahimi Tabar, A. R. Rastegar, and F. Shahbazi, Phys. Rev. E **63**, 056308 (2001).
[86] B. Derrida and J. L. Lebowitz, Phys. Rev. Lett. **80**, 209 (1998).
[87] B. Derrida and C. Appert, J. Stat. Phys. **94**, 1 (1999).
[88] J. M. Kim, M. A. Moore, and A. J. Bray, Phys. Rev. A **44**, 2345 (1991).
[89] A. A. Masoudi, F. Shahbazi, J. Davoudi, and M. R. Rahimi Tabar, in preparation.
[90] J. Bec, e-print nlin.CD/0103029.
[91] M. Plischke *et al.*, Phys. Rev. A **35**, 3485 (1987).
[92] J. Neergard and M. Den Nijs, J. Phys. A **30**, 1935 (1997).
[93] J. Bec (private communication).
[94] M. Prahofer and H. Spohn, Phys. Rev. Lett. **84**, 4882 (2000); e-print cond-mat/0101200; e-print math.PR/0105240.